Zadatak 1.

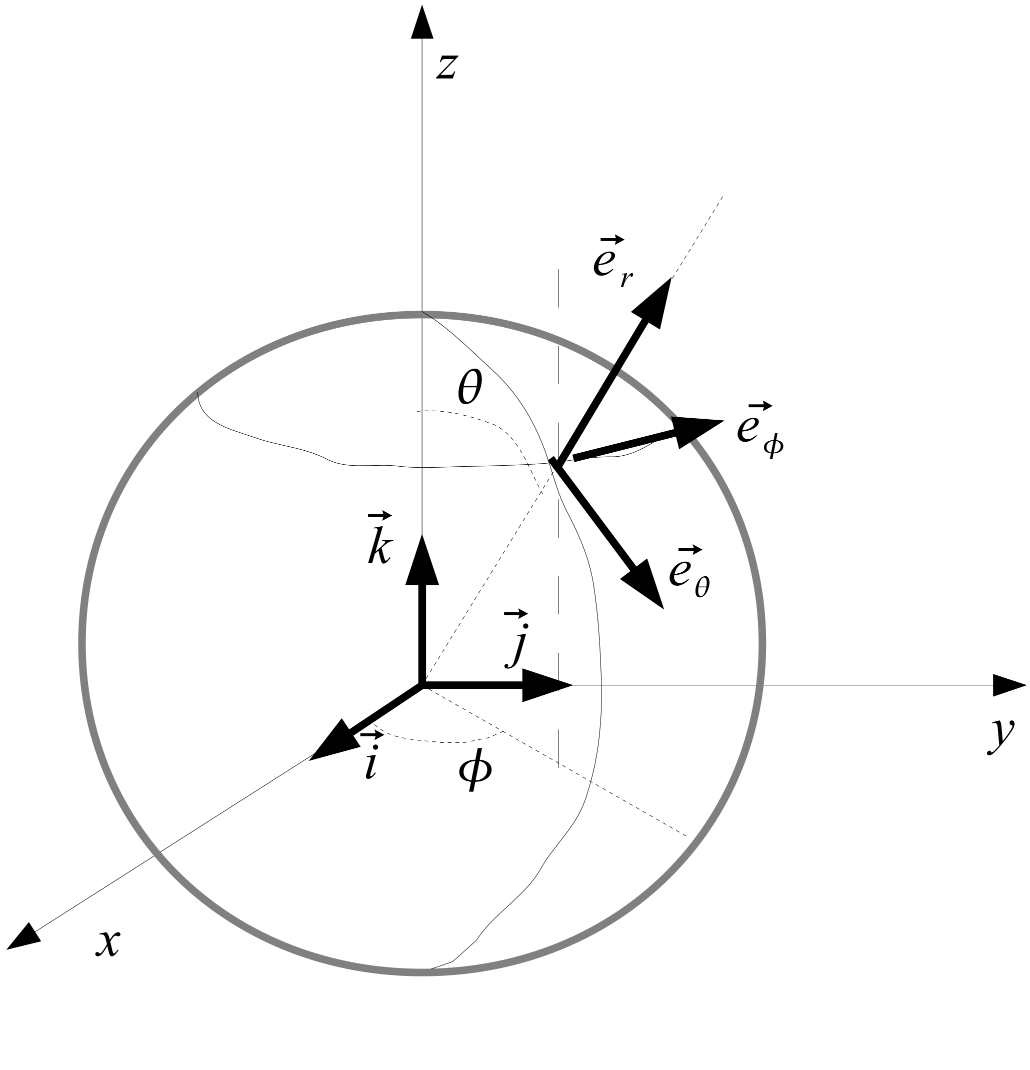
Napišite operator gradijenta u sfernim koordinatama.

Rješenje:

U kartezijevim pravokutnim koordinatama operator gradijenta je jednak

$$\vec{\nabla} = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Sferni se koordinatni sustav satoji od tri koordinate r, θ, ϕ kojima moramo pridružiti tri jedinična vektora $\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$



Odnos između koordinata i jediničnih vektora kartezijevog i sfernog sustava je

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

$$\vec{e}_r = \vec{i}\sin(\theta)\cos(\phi) + \vec{j}\sin(\theta)\sin(\phi) + \vec{k}\cos(\theta)$$
$$\vec{e}_\theta = \vec{i}\cos(\theta)\cos(\phi) + \vec{j}\cos(\theta)\sin(\phi) - \vec{k}\sin(\theta)$$
$$\vec{e}_\phi = -\vec{i}\sin(\phi) + \vec{j}\cos(\phi)$$

$$\vec{i} = \vec{e}_r\sin(\theta)\cos(\phi) + \vec{e}_\theta\cos(\theta)\cos(\phi) - \vec{e}_\phi\sin(\phi)$$
$$\vec{j} = \vec{e}_r\sin(\theta)\sin(\phi) + \vec{e}_\theta\cos(\theta)\sin(\phi) + \vec{e}_\phi\cos(\phi)$$
$$\vec{k} = \vec{e}_r\cos(\theta) - \vec{e}_\theta\sin(\theta)$$

Sada još treba izraziti derivacije po *x, y, z* s pomoću derivacija po r, θ, ϕ . Imamo

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\sin(\theta)\cos(\phi),\ \frac{\partial r}{\partial y}=\frac{y}{r}=\sin(\theta)\sin(\phi),\ \frac{\partial r}{\partial z}=\frac{z}{r}=\cos(\theta)$$

$$\frac{\partial \theta}{\partial x}=\frac{\cos(\theta)\cos(\phi)}{r}\ ,\ \frac{\partial \theta}{\partial y}=\frac{\cos(\theta)\sin(\phi)}{r}\ ,\ \frac{\partial \theta}{\partial z}=-\frac{\sin(\theta)}{r}$$

$$\frac{\partial \phi}{\partial x}=-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\ ,\ \frac{\partial \phi}{\partial y}=\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\ ,\ \frac{\partial \phi}{\partial z}=0$$

Sada imamo

$$\frac{\partial}{\partial x}=\sin(\theta)\cos(\phi)\frac{\partial}{\partial r}+\frac{\cos(\theta)\cos(\phi)}{r}\frac{\partial}{\partial \theta}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y}=\sin(\theta)\sin(\phi)\frac{\partial}{\partial r}+\frac{\cos(\theta)\sin(\phi)}{r}\frac{\partial}{\partial \theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z}=\cos(\theta)\frac{\partial}{\partial r}-\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}$$

Konačni rezultat (detalje prikazati na ploči):

$$\vec{\nabla}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\vec{e}_\phi\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}$$

Zadatak 2.

Napišite operator zamaha $\vec{L}=\vec{r}\times\vec{p}$ u kartezijevim i sfernim koordinatama.

Rješenje:

Po zakonu vektorskog množenja imamo:

$$L_x=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)=$$

$$=-i\hbar\left[r\sin(\theta)\sin(\phi)\left(\cos(\theta)\frac{\partial}{\partial r}-\frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right)-\right.$$

$$\left.-r\cos(\theta)\left(\sin(\theta)\sin(\phi)\frac{\partial}{\partial r}+\frac{\cos(\theta)\sin(\phi)}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)\right]=$$

$$=-i\hbar\left[-\sin(\phi)\frac{\partial}{\partial\theta}-\cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right]=$$

$$=i\hbar\left[\sin(\phi)\frac{\partial}{\partial\theta}+\cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right]$$

$$L_y=-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)=$$

$$=i\hbar\left(-\cos(\phi)\frac{\partial}{\partial\theta}+\cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi}\right)$$

$$L_z=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\hbar\frac{\partial}{\partial\phi}$$

Zadatak 3.
Napišite operatore $L_{\pm} = L_x \pm i L_y, L_z$ u sfernim koordinatama i izvedite njihova komutacijska pravila.
Rješenje:

$$L_{\pm} = i\hbar \mathrm{e}^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot(\theta)\frac{\partial}{\partial\phi}\right) \quad , \quad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\left[L_z, L_{\pm}\right] = \hbar^2\left[\frac{\partial}{\partial\phi}, \mathrm{e}^{\pm i\phi}\right]\frac{\partial}{\partial\theta} \pm i\hbar^2\left[\frac{\partial}{\partial\phi}, \mathrm{e}^{\pm i\phi}\right]\cot(\theta)\frac{\partial}{\partial\phi} = \pm\hbar L_{\pm}$$

$$\left[L_+, L_-\right] = 2\hbar L_z$$

Zadatak 4.
Dokažite da operator

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$

komutira s L_x, L_x i L_z

Zadatak 5.
Izrazite operator $\vec{p}^2$ u sfernim koordinatama i povežite ga s operatorom $\vec{L}^2$.

Zadatak 6.
Napišite Schrödingerovu jednadžbu za česticu koja se giba po kružnici.

Rješenje:
Gibanje po kružnici je ravninsko gibanje, koje opisujemo dvjema prostornim koordinatama x i y. Zbog toga je pogodno uzeti cilindrične koordinate

$$x = r\cos(\phi) \quad , \quad y = r\sin(\phi)$$

i izraziti operator $\vec{p}^{\,2}$ s pomoću koordinata ρ, ϕ. Imamo

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}$$

Kvadrirajmo ove operatore:

$$\frac{\partial^2}{\partial x^2} = \cos^2(\phi)\frac{\partial^2}{\partial r^2} - \cos(\phi)\sin(\phi)\left(-\frac{1}{r^2}\frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial \phi \partial r}\right) +$$
$$+ \frac{\sin^2(\phi)}{r}\frac{\partial}{\partial r} - \frac{\sin(\phi)\cos(\phi)}{r}\frac{\partial^2}{\partial \phi \partial r} + \frac{\sin(\phi)\cos(\phi)}{r^2}\frac{\partial}{\partial \phi} +$$
$$+ \frac{\sin^2(\phi)}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2(\phi)\frac{\partial^2}{\partial r^2} + \cos(\phi)\sin(\phi)\left(-\frac{1}{r^2}\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi\,\partial r}\right) +$$
$$+\frac{\cos^2(\phi)}{r}\frac{\partial}{\partial r} + \frac{\sin(\phi)\cos(\phi)}{r}\frac{\partial^2}{\partial\phi\,\partial r} - \frac{\sin(\phi)\cos(\phi)}{r^2}\frac{\partial}{\partial\phi} +$$
$$+\frac{\cos^2(\phi)}{r^2}\frac{\partial^2}{\partial\phi^2}$$

Zbrajanjem dviju prethodnih jednadžbi dobivamo

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$$

Gibanje po kružnici znači da valna funkcija ne ovisi o *r*. Prema tome Schrödingerova jednadžba je

$$-\frac{\hbar^2}{2mr^2}\frac{\partial^2}{\partial\phi^2}\psi(\phi) = E\,\psi(\phi)$$

No, derivacija po kutu je zapravo operator zamaha, pa imamo

$$\frac{L_z^2}{2I}\psi = E\,\psi$$

gdje je *I* moment tromosti čestice.