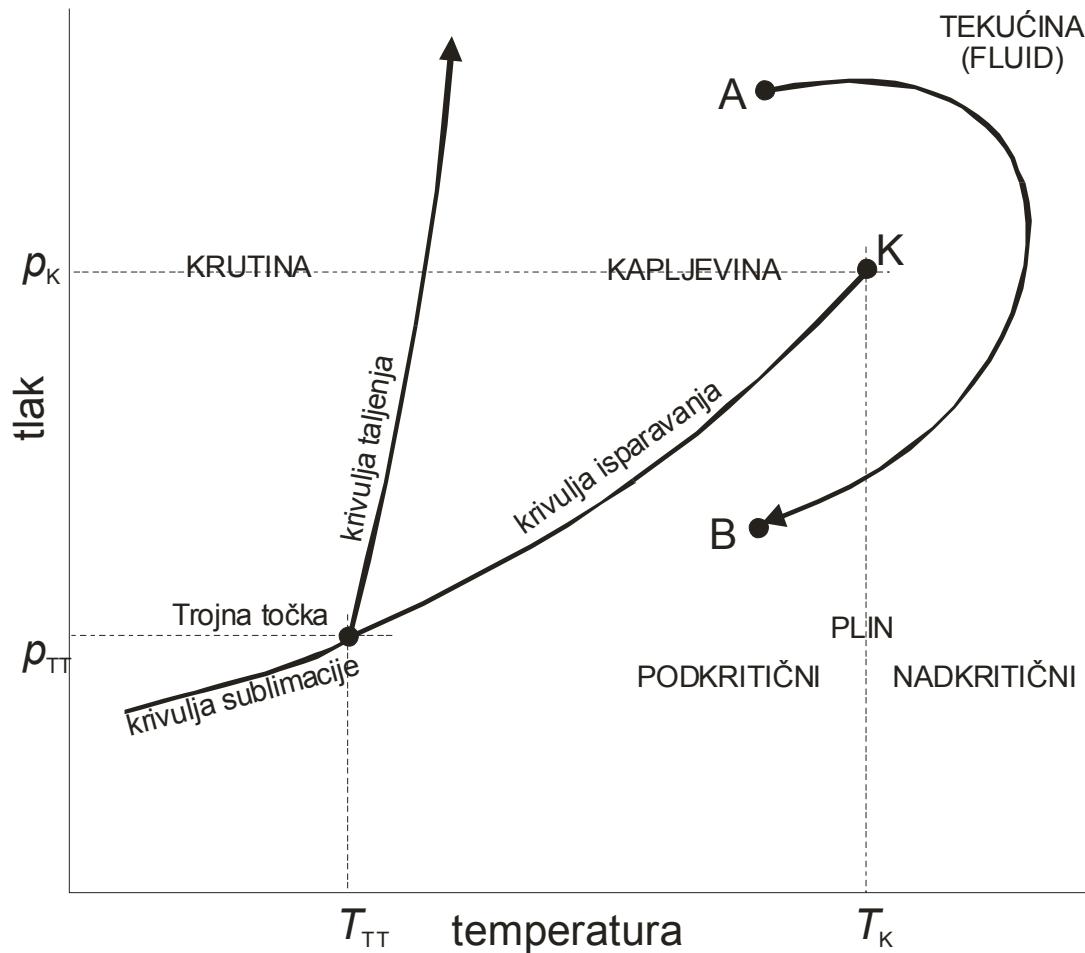
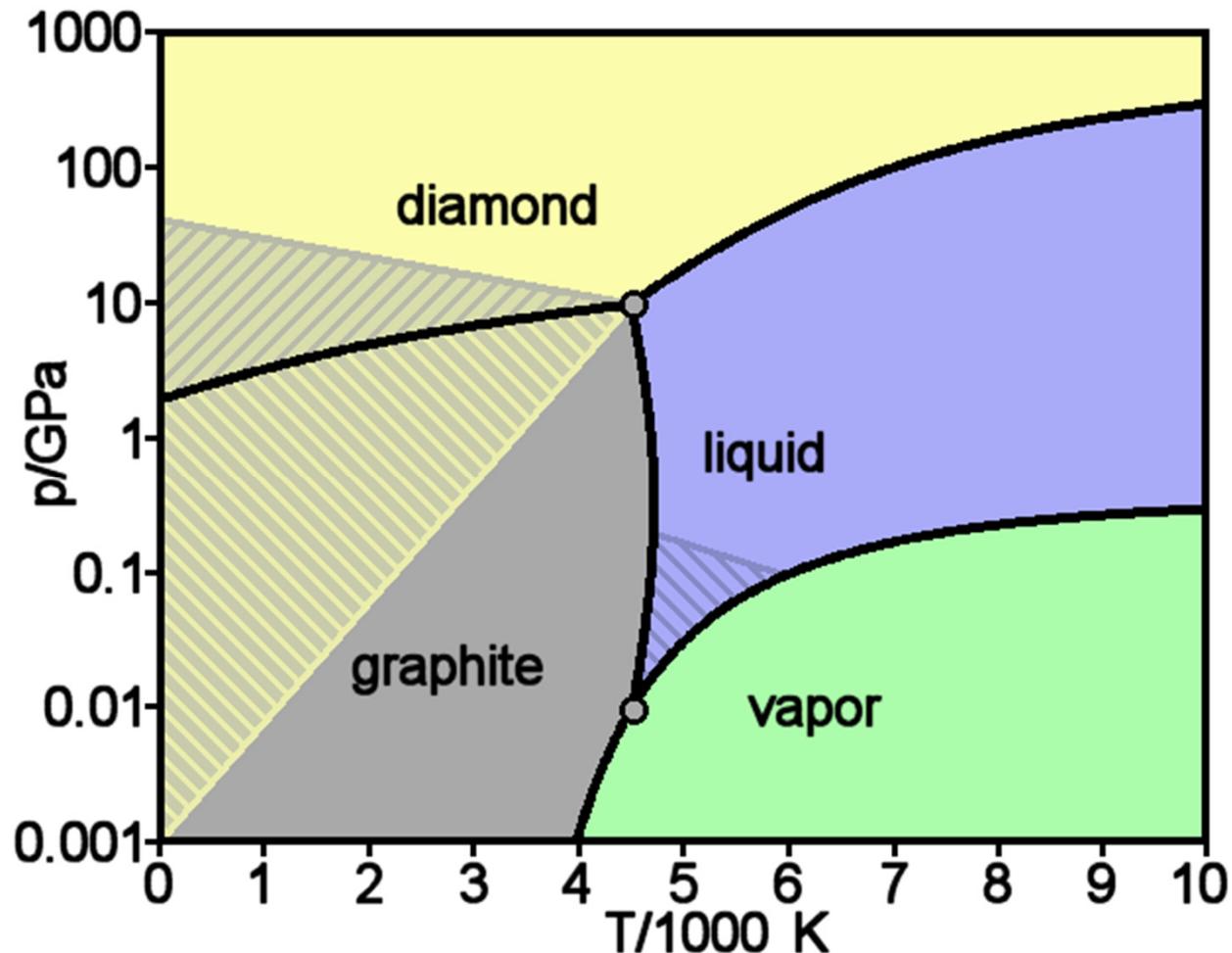


Volumetrijska svojstva realnih fluida

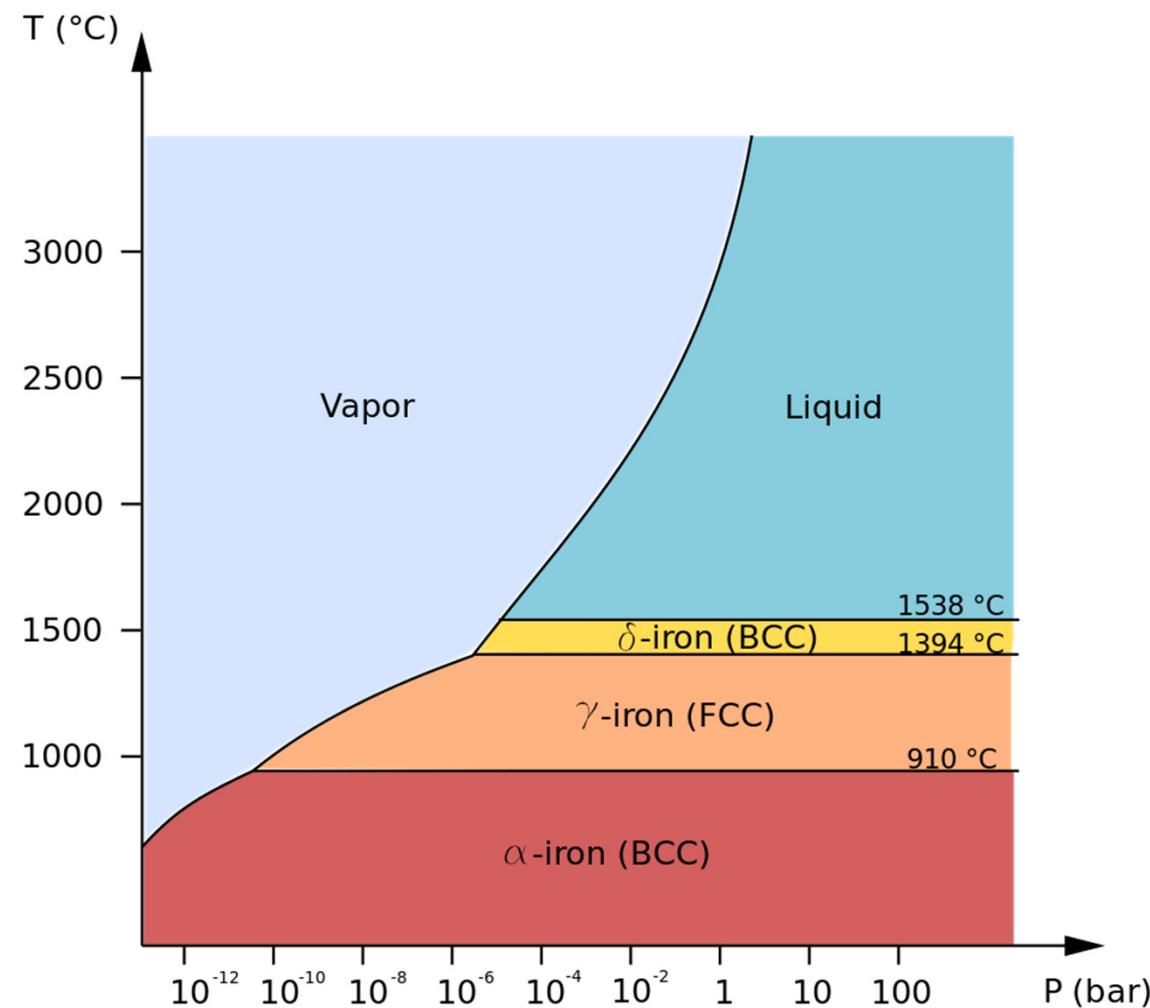
p - T dijagram za čistu tvar



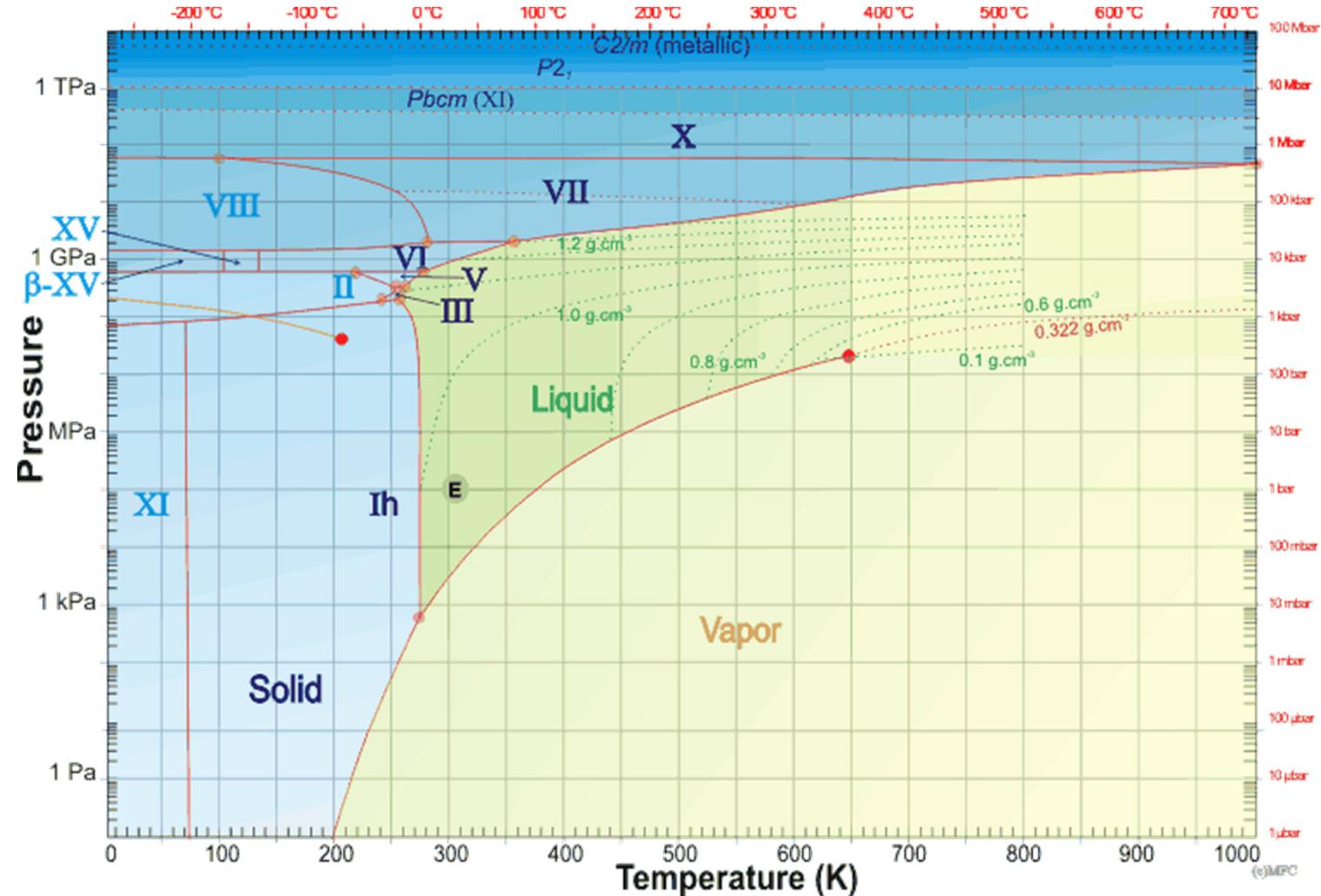
p - T dijagram za čistu tvar



p - T (T - p)dijagram za čistu tvar



p - T dijagram za čistu tvar



Jednadžbe stanja

Gibbsovo pravilo faza

$$f = N_K - N_F + 2$$

Jednofazni, jednokomponentni sustavi

$$f = 2$$

Termodinamička svojstva

$$p, T, s, v, h, u, g, \dots$$

Jednadžbe stanja u širem smislu:

$$\begin{aligned} p &= f(T, s) & T &= f(s, v) \\ g &= f(u, v) & h &= f(p, T) \end{aligned}$$

Jednadžbe stanja u užem smislu:

$$p = f(v, T)$$

$$f(p, T, v) = 0 \quad v = f(p, T)$$

Opća plinska jednadžba

Robert Boyle i Edme Mariotte

$$(pV)_T = \text{konst}$$

Jacques Charles, Joseph Louis Gay-Lussac

$$\left(\frac{V}{T}\right)_p = \text{konst}$$

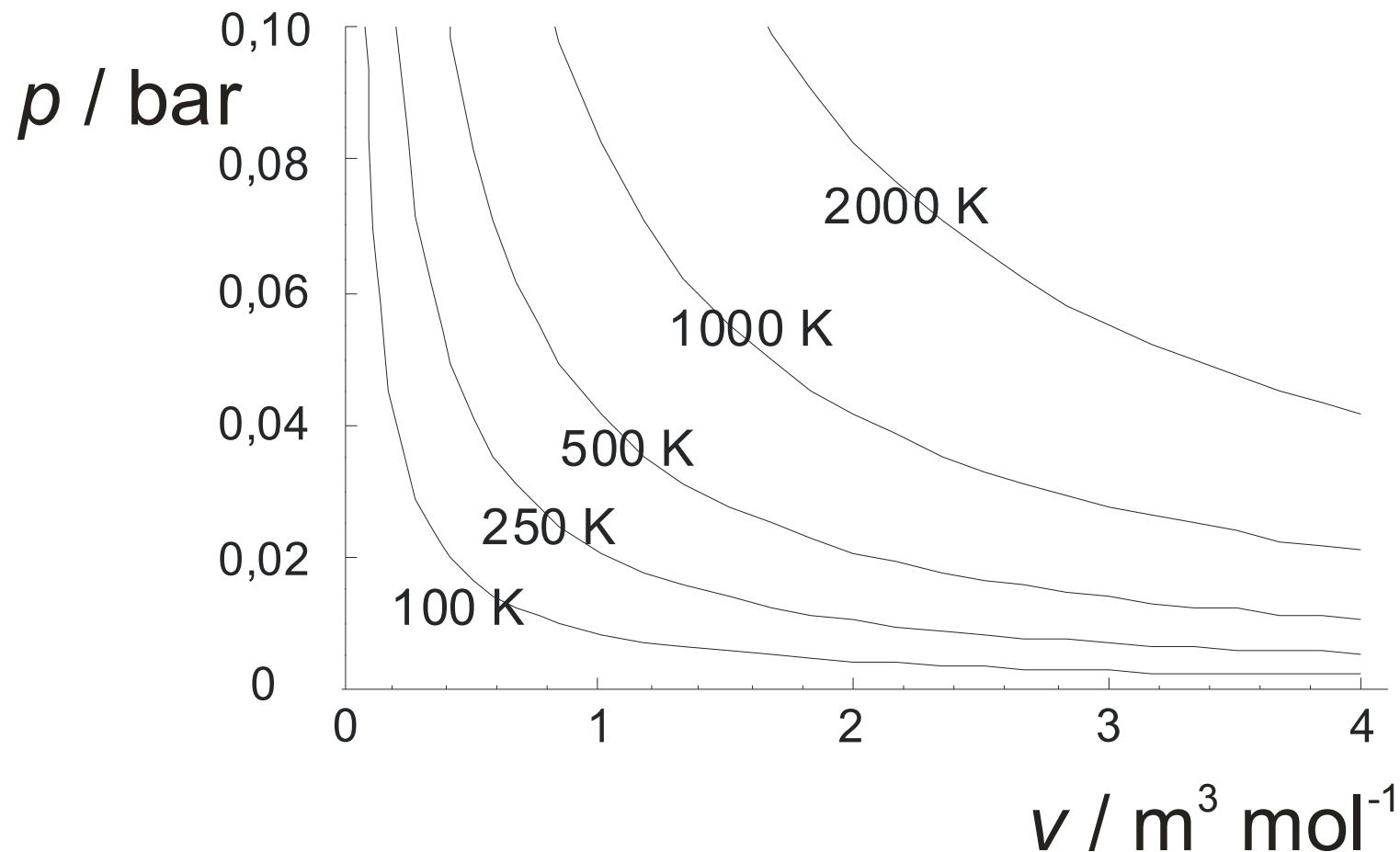
Avogadro:

Jednaki volumeni plina sadrže jednak broj čestica

$$pV = nRT$$

$$pV = RT$$

Opća plinska jednadžba



Opća plinska jednadžba

Izvod: metodama statističke termodinamike iz modela idealnog plina:

Pretpostavke:

zanemariv volumen čestica
zanemariva međudjelovanja
(elastični srazovi)

Uvjeti:

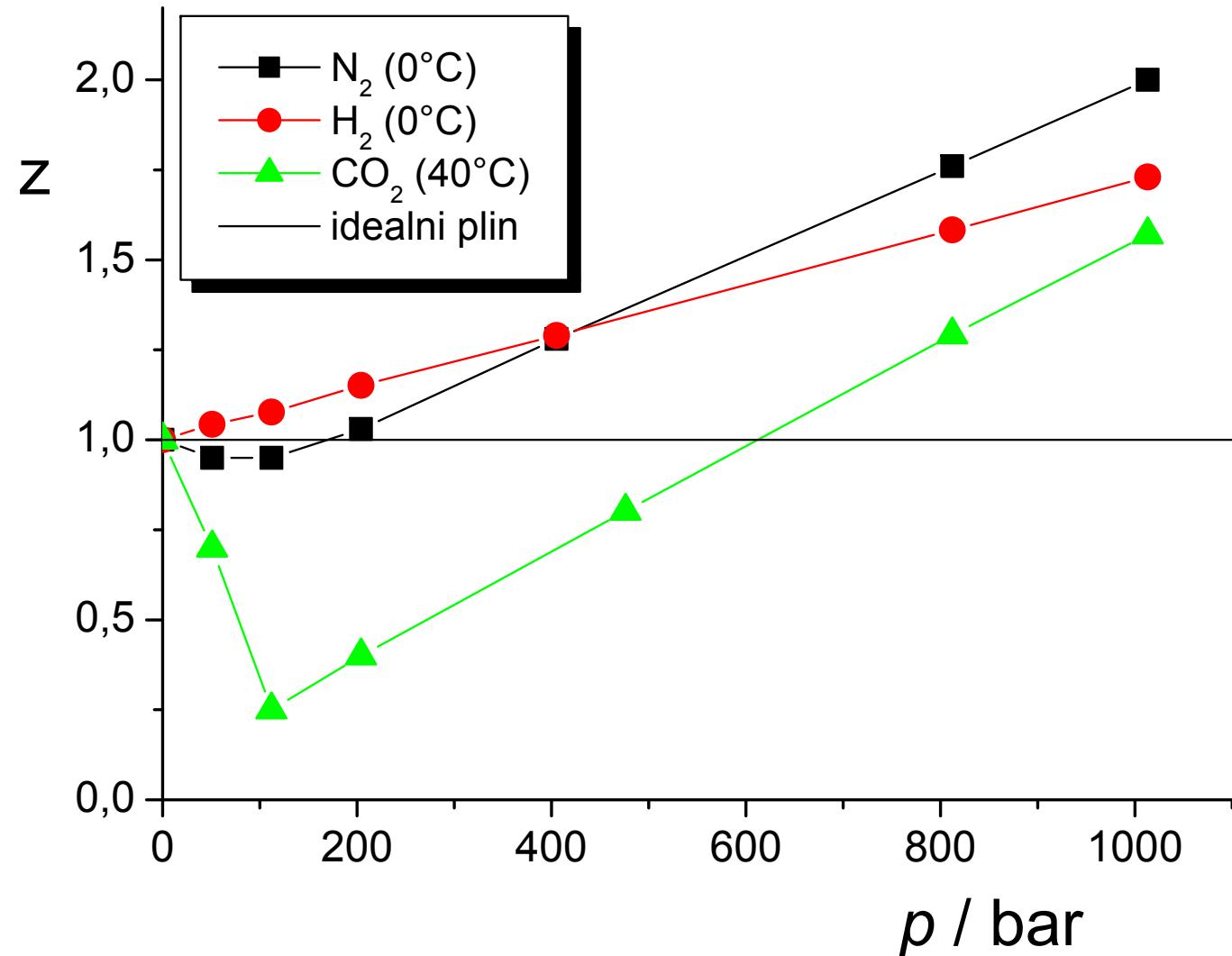
visoke temperature
niski tlakovi

Odstupanja od idealnosti

Volumetrijski podaci

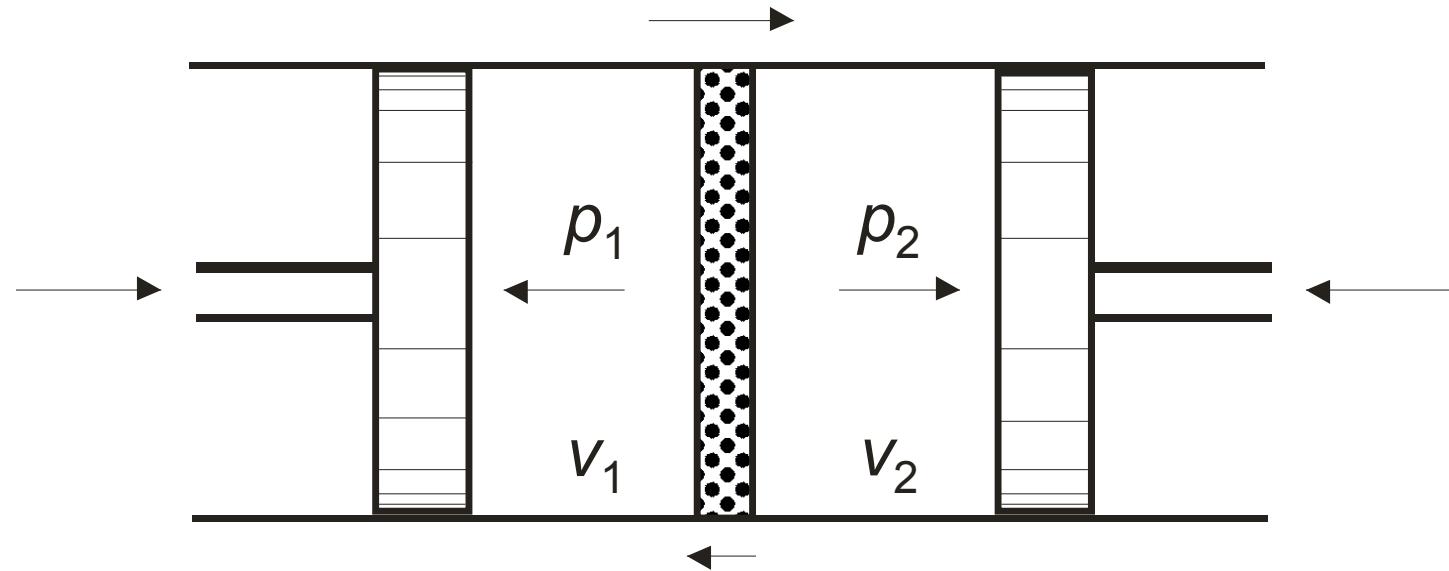
$$\lim_{p \rightarrow 0} (pV) = RT$$

$$z = \frac{pV}{RT} = \frac{v}{v_{id}}$$



Odstupanja od idealnosti

Prigušivanje
realnog
plina
Joule, Kelvin



Uvjet stalne entalpije
(nema izmjene topline s okolinom)

Odstupanja od idealnosti

Prigušivanje realnog plina

Totalni diferencijal:

$$dh = \left(\frac{\partial h}{\partial p} \right)_T dp + \left(\frac{\partial h}{\partial T} \right)_p dT$$

$$dh=0 \quad \left(\frac{\partial h}{\partial T} \right)_p = c_p$$

Joule -Thomsonov koeficijent:

$$\mu = \left(\frac{\partial T}{\partial p} \right)_h = -\frac{1}{c_p} \left(\frac{\partial h}{\partial p} \right)_T$$

Ovisnost entalpije plina o tlaku:

$$\left(\frac{\partial h}{\partial p} \right)_T = v - T \left(\frac{\partial v}{\partial T} \right)_p$$

Ovisnost entalpije idealnog plina o tlaku:

$$v=RT/p, (\partial v/\partial T)p=R/p$$

$$\left(\frac{\partial h}{\partial p} \right)_T = \frac{RT}{p} - T \frac{R}{p} = 0$$

Odstupanja od idealnosti

Ukapljivanje realnog plina

Kritična temperatura

najviša temperatura na kojoj se plin još može ukapljiti

Kritični tlak

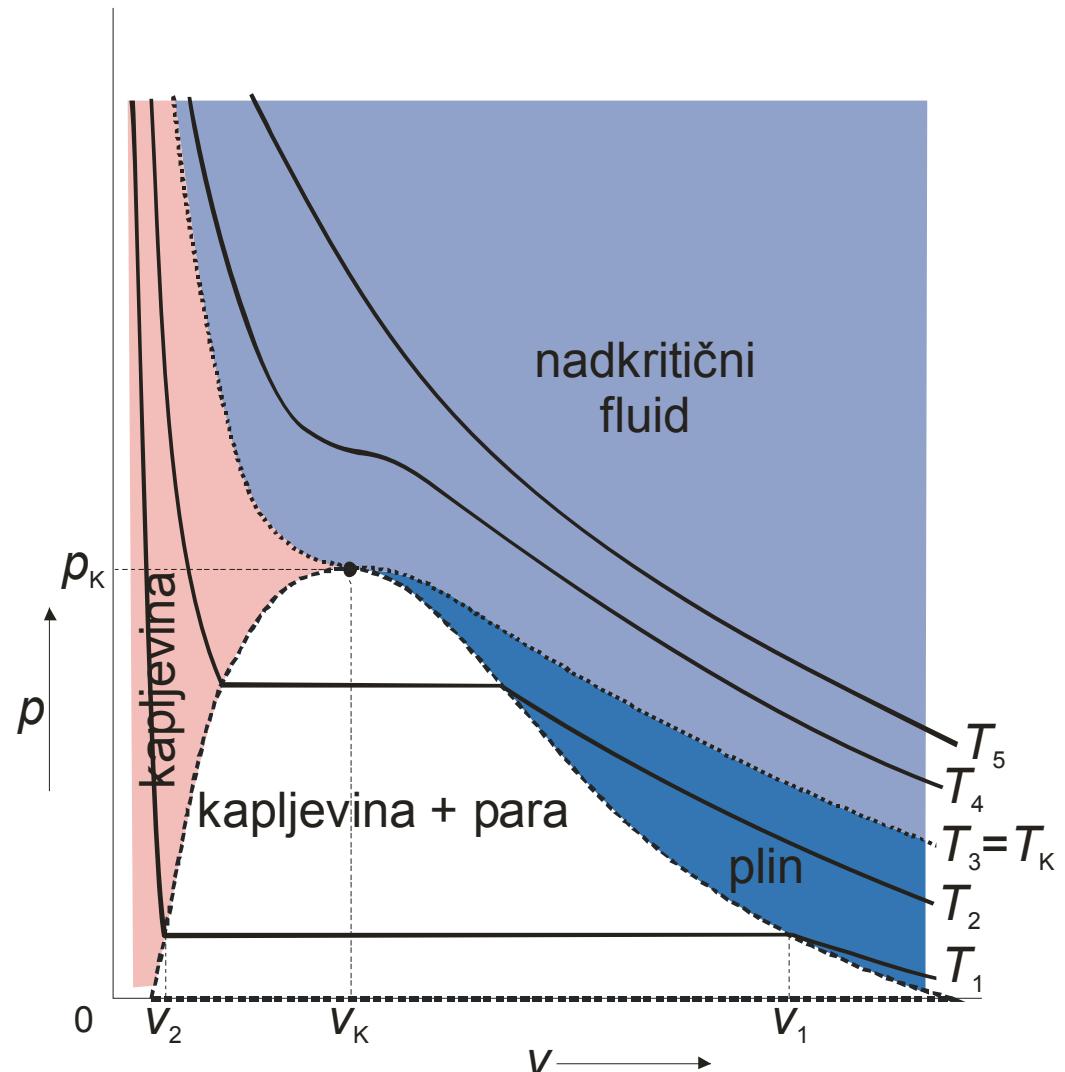
tlak potreban za ukapljivanje plina pri kritičnoj temperaturi

Kritični molarni volumen

volumen 1 mola plina pri kritičnoj temperaturi i kritičnom tlaku

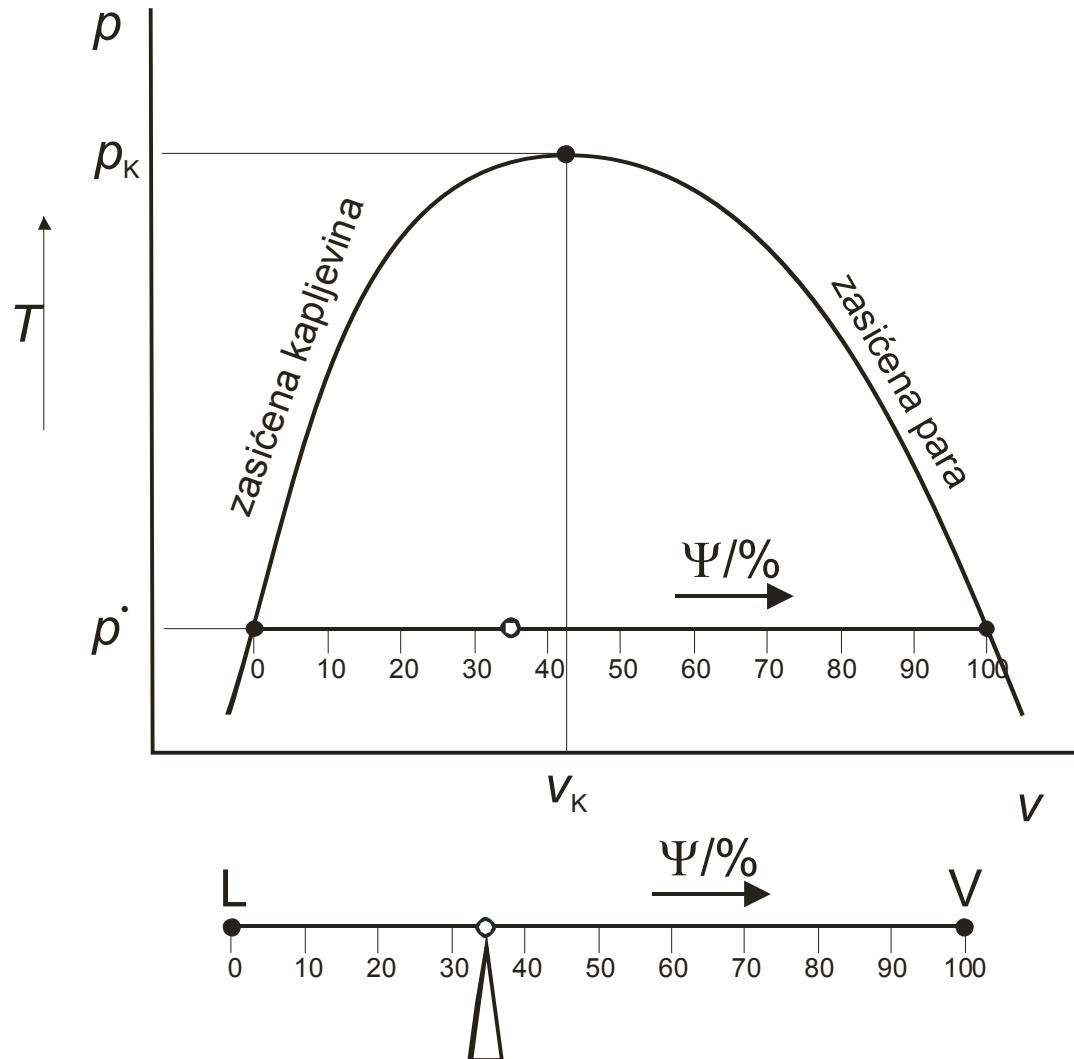
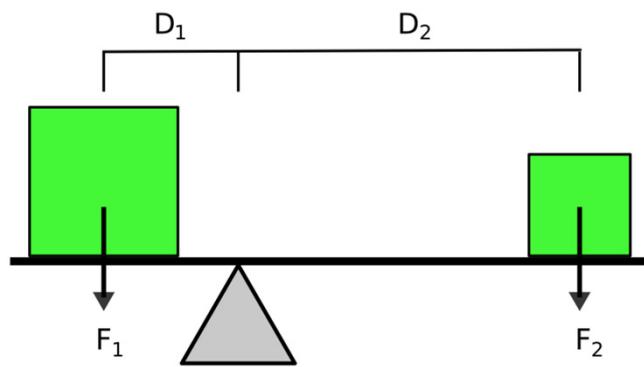
Kritična gustoća

gustoća plina pri kritičnom tlaku i kritičnoj temperaturi



Odstupanja od idealnosti

Pravilo poluge



Virijalna jednadžba stanja

Opis međudjelovanja

$$\Gamma = \Gamma_{\text{odb}} + \Gamma_{\text{priv}} = \frac{A}{r^n} - \frac{B}{r^m}$$

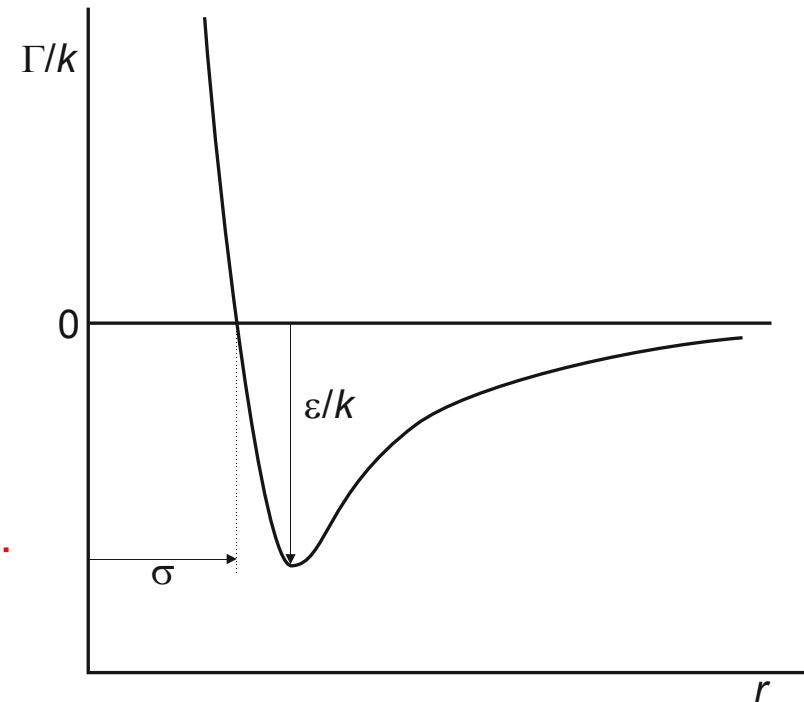
van der Waalsove sile:

inducirani dipol – inducirani dipol (London)

dipol – inducirani dipol (Debye)

dipol – dipol (Keesom)

Coulombova elektrostatska međudjelovanja.



Virijalna jednadžba stanja

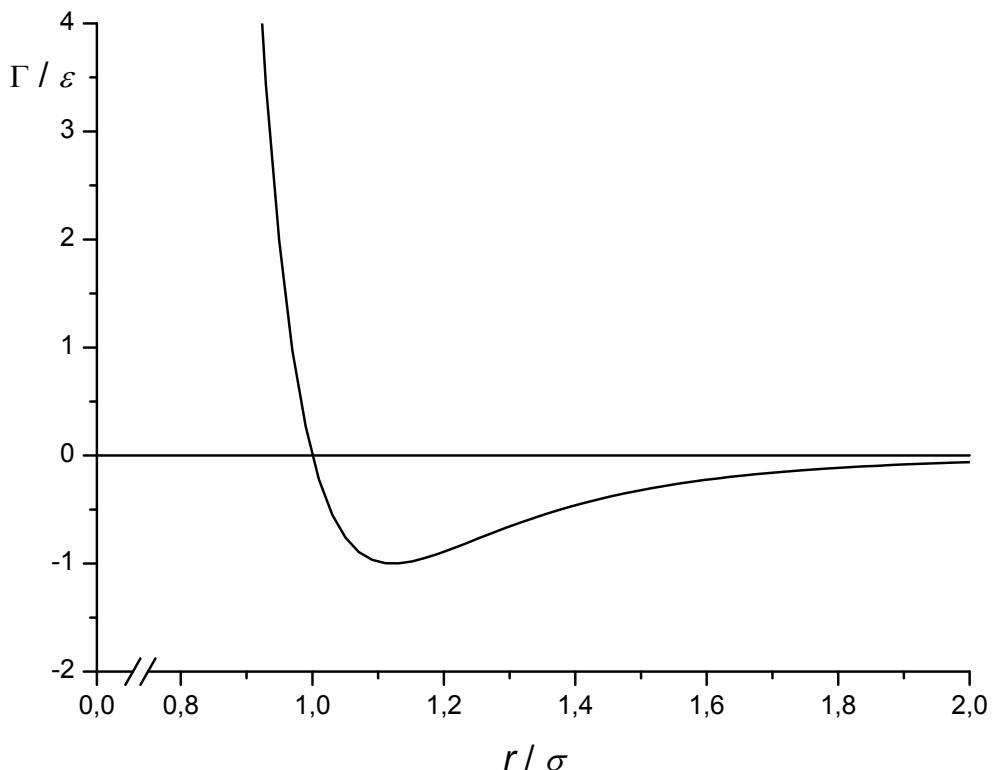
Lennard Jones 6 -12
potencijal

$$\Gamma = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right]$$

Clausiusov
(virijalni poučak)

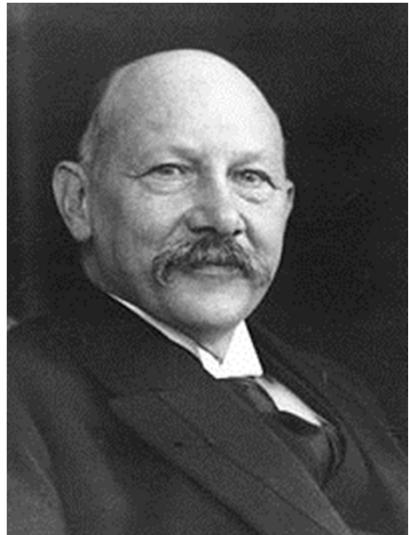
$$z = \frac{pv}{RT} = 1$$

$$z = \frac{pv}{RT} = 1 + \frac{B(T)}{v}$$



Virijalna jednadžba stanja

Heike
Kamerlingh Onnes
(Nobel 1913.)



Proširena, empirijska virijalna jednadžba stanja

$$z = \frac{pv}{RT} = 1 + \frac{B(T)}{v} + \frac{C(T)}{v^2} + \dots$$

$$z = \frac{pv}{RT} = 1 + B(T) \frac{p}{RT} + \left(C(T) - B(T)^2 \right) \left(\frac{p}{RT} \right)^2 + \dots$$

Virijalna jednadžba stanja

Pojednostavljenja

$$\frac{pv}{RT} = 1 + \frac{B(T)}{RT} p \quad p[v - B(T)] = RT$$

$$v > 2v_K$$

$$\frac{pv}{RT} = 1 + \frac{B(T)}{v} + \frac{C(T)}{v^2} \quad v^3 - \frac{RT}{p} v^2 - \frac{B(T)RT}{p} v - \frac{C(T)RT}{p} = 0$$

do 50 bar, POLINOM TREĆEG STUPNJA PO VOLUMENU

Virijalna jednadžba stanja

Empirijska proširenja

Benedict, Webb i Rubin (BWR), 1940

$$z = \frac{pv}{RT} = 1 + \frac{B}{v} + \frac{C}{v^2} + \frac{a\alpha}{RTv^5} + \frac{\beta}{RT^3 v} \left(1 + \frac{\gamma}{v^2} \right) \exp \left(-\frac{\gamma}{v^2} \right)$$
$$B = \left(b_1 - \frac{b_2}{RT} - \frac{b_3}{RT^3} \right) \quad C = \left(c_1 - \frac{c_2}{RT} \right)$$

Starling BWR (SBWR), 1973 REDUCIRANI OBLIK

$$z = \frac{pv}{RT} = 1 + \frac{B}{v_r} + \frac{C}{v_r^2} + \frac{D}{v_r^5} + \frac{c_4}{T_r^3 v_r^2} + \left[\beta + \frac{\gamma}{v_r^2} \right] \exp \left[-\frac{\gamma}{v_r^2} \right]$$

$$B = b_1 - \frac{b_2}{T_r} - \frac{b_3}{T_r^2} - \frac{b_4}{T_r^3} \quad C = c_1 - \frac{c_2}{T_r} - \frac{c_3}{T_r^2} \quad D = d_1 - \frac{d_2}{T_r}$$

vdW jednadžba stanja

Johannes Diderik van der Waals 1873

$$p = \frac{RT}{v-b} - \frac{a}{v^2} \quad v^3 - v^2 \left(b + \frac{RT}{p} \right) + v \left(\frac{a}{p} \right) - \frac{ab}{p} = 0$$

Usporedba s
virijalnom
jednadžbom

$$v^3 - \frac{RT}{p} v^2 - \frac{B(T)RT}{p} v - \frac{C(T)RT}{p} = 0$$

POLINOM TREĆEG STUPNJA PO VOLUMENU

Usporedba s
općom plinskom
jednadžbom

$$\left(p + \frac{a}{v^2} \right) (v - b) = RT$$

vdW jednadžba stanja

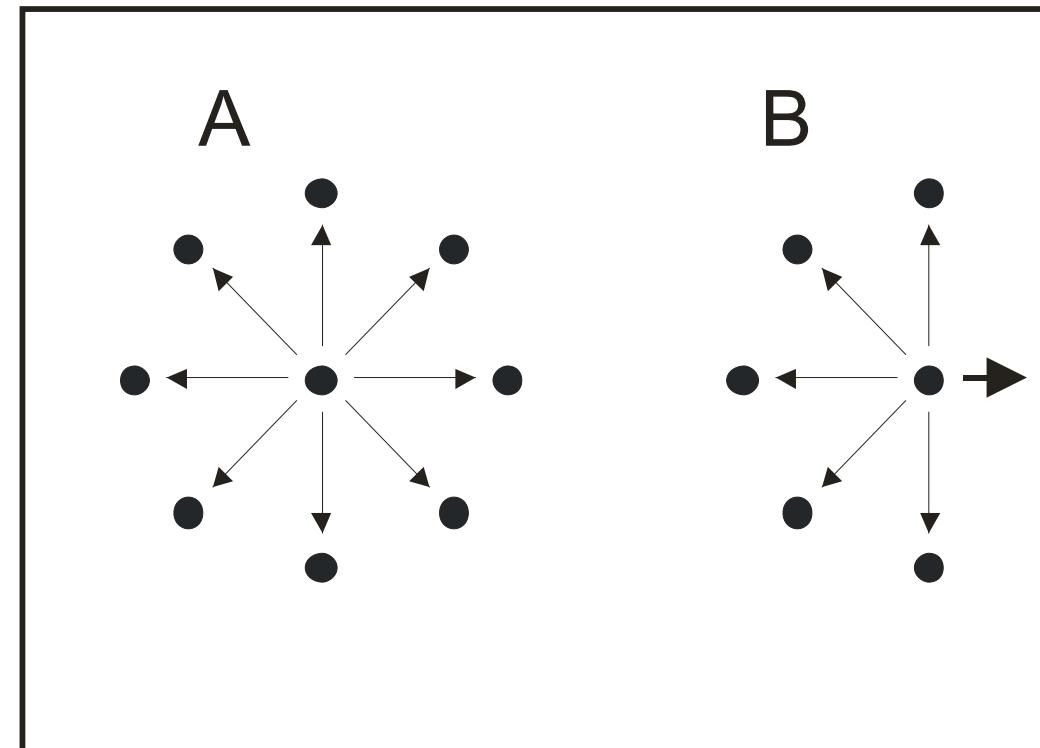
Korekcija volumena

$$(v - b)$$

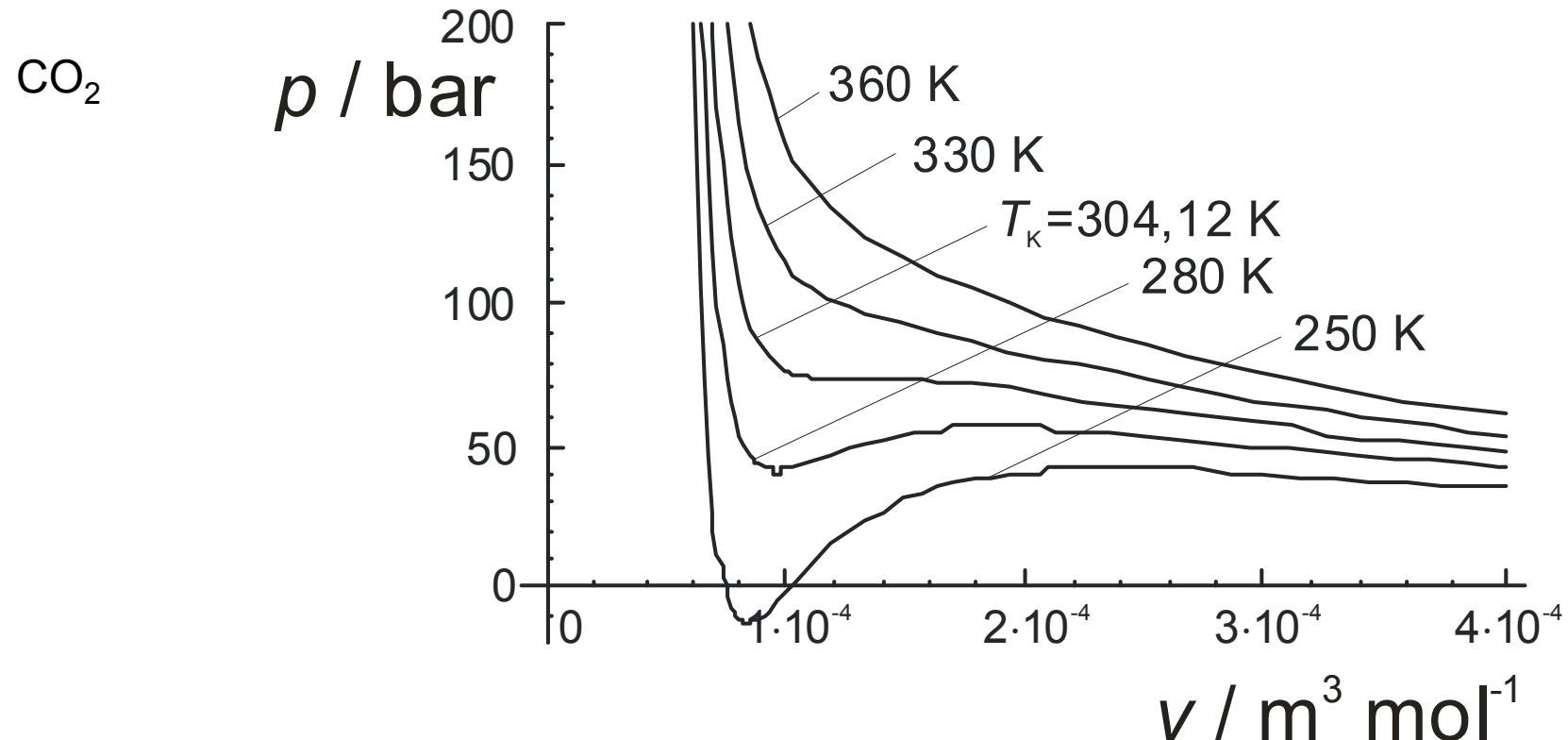


Korekcija tlaka

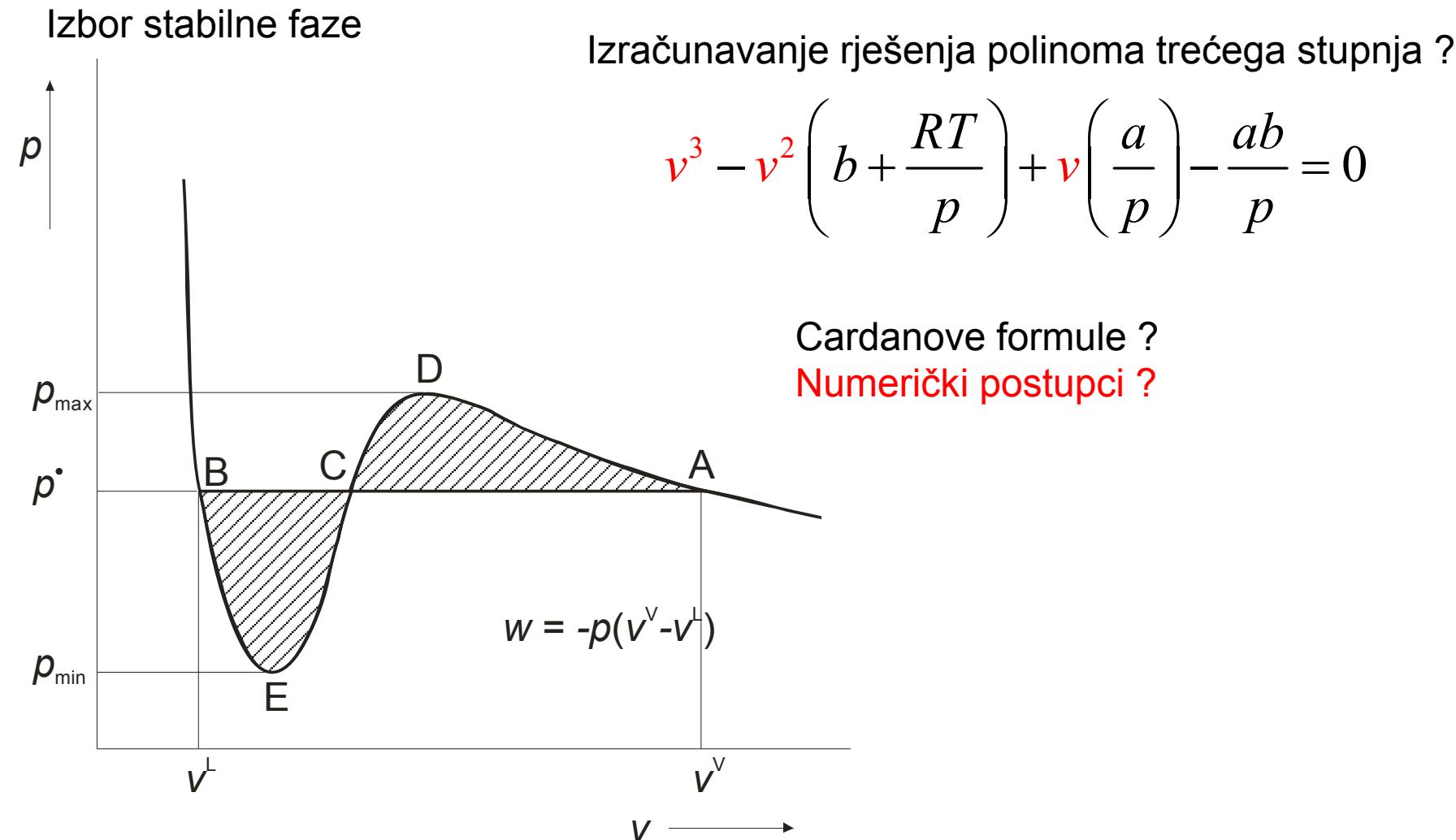
$$\left(p + \frac{a}{v^2} \right)$$



vdW jednadžba stanja



vdW jednadžba stanja



vdW jednadžba stanja

Izbor stabilne faze

$$w_{\text{real}} = - \int_{v^L}^{v^V} p dv = -p(v^V - v^L)$$

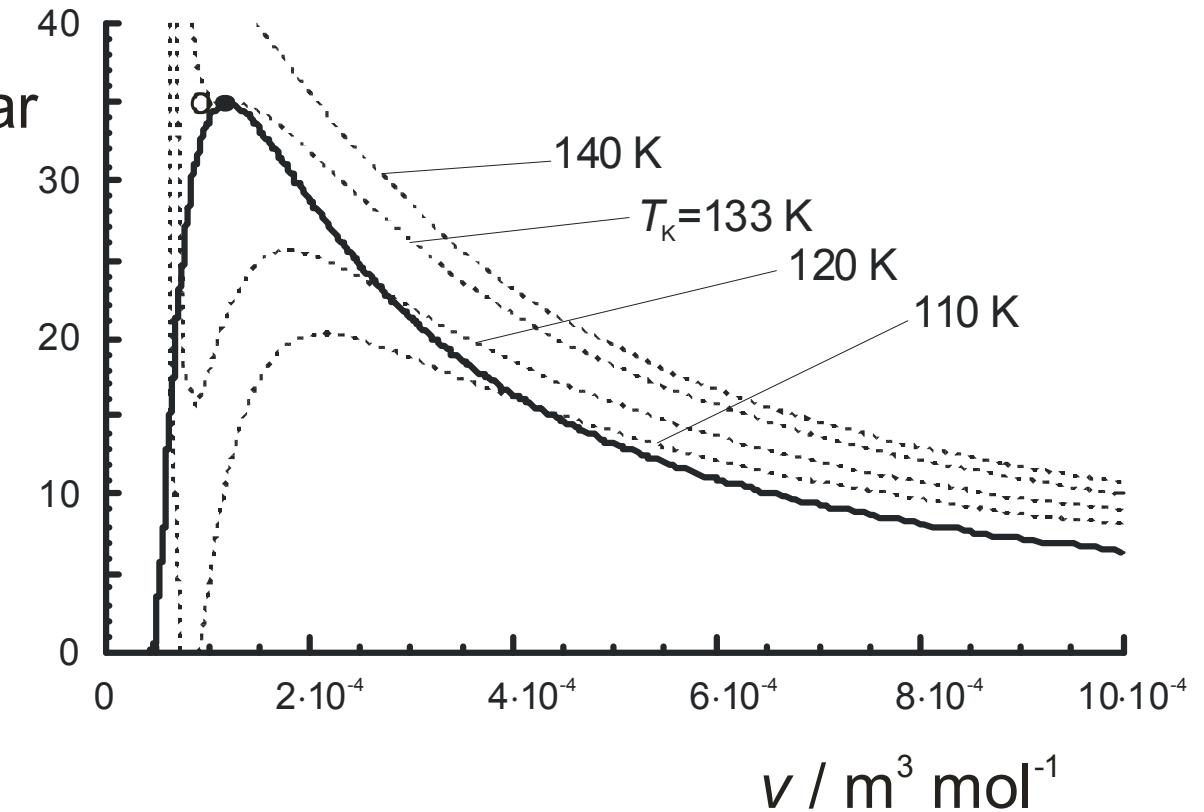
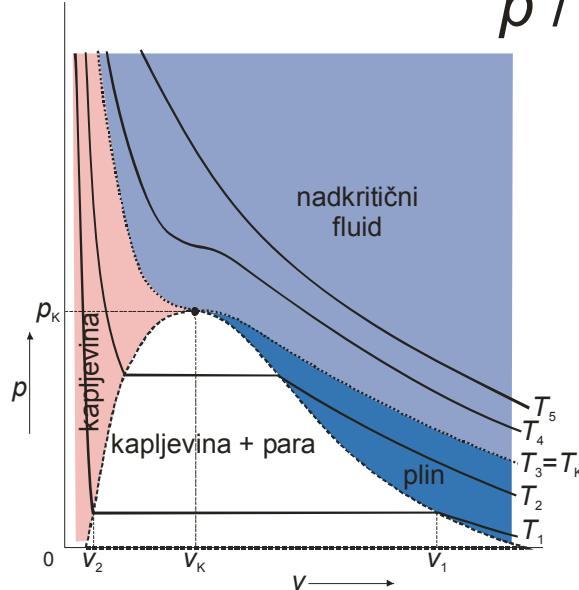
$$w_{\text{vdW}} = - \int_{v^L}^{v^V} p(v) dv = - \int_{v^L}^{v^V} \left(\frac{RT}{v-b} - \frac{a}{v^2} \right) dv = - \left(RT \ln \frac{v^V - b}{v^L - b} - \frac{v^V - v^L}{v^V v^L} a \right)$$

$$p(v^V - v^L) = RT \ln \frac{v^V - b}{v^L - b} - \frac{v^V - v^L}{v^V v^L} a$$

vdW jednadžba stanja

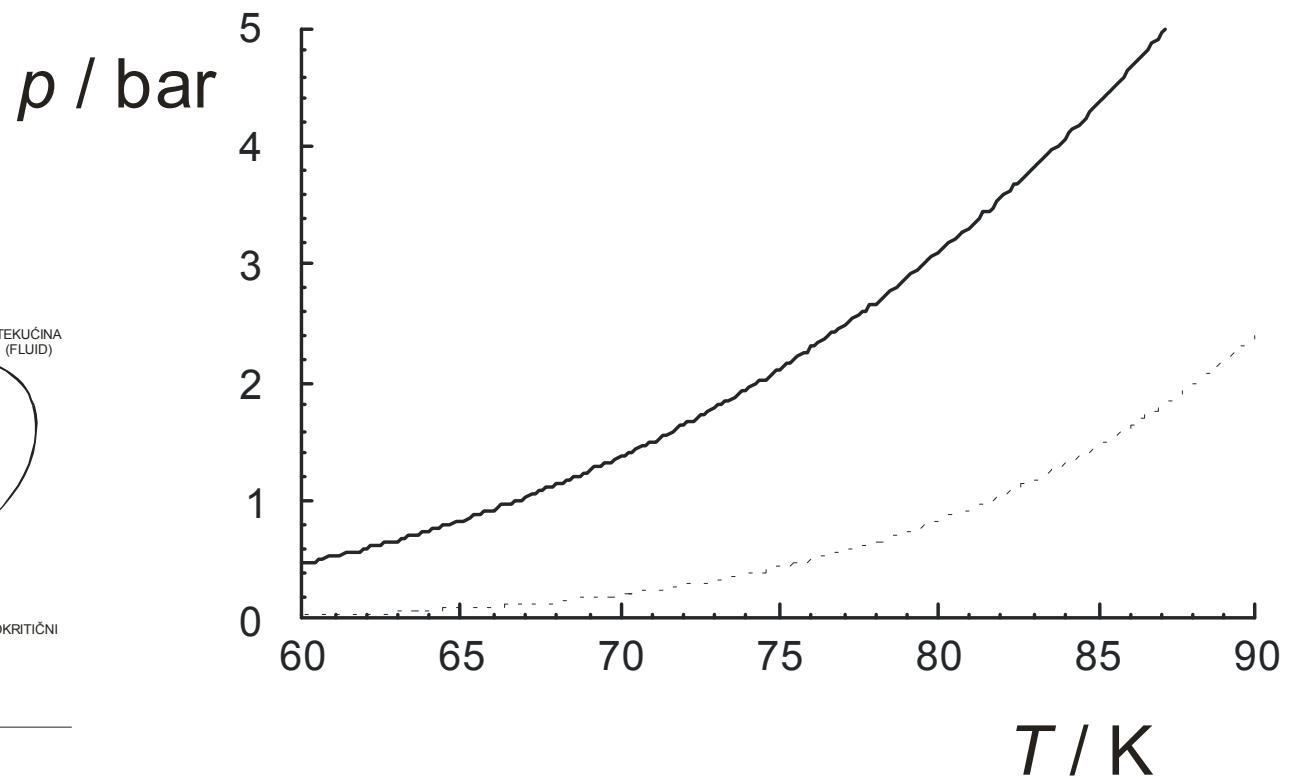
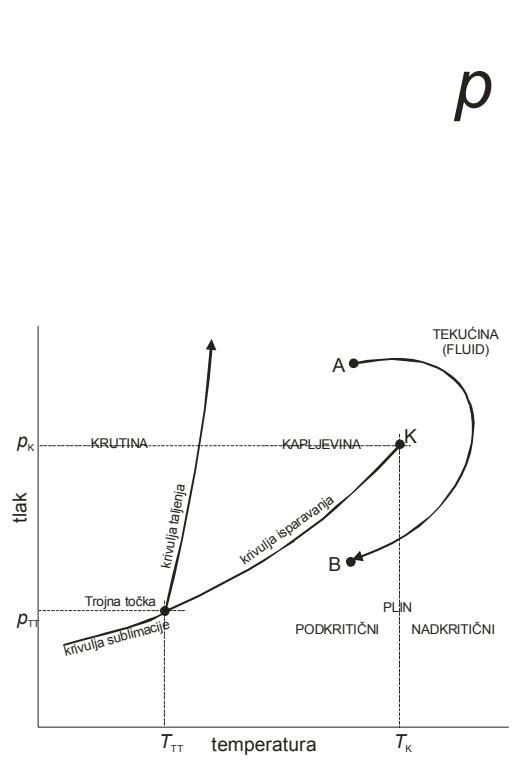
Izračunavanje ravnotežnog tlaka, p^\bullet

$$p = \frac{RT}{v^V - v^L} \ln \frac{v^V - b}{v^L - b} - \frac{a}{v^V v^L}$$



vdW jednadžba stanja

Izračunavanje ravnotežnog tlaka



vdW parametri

Troparametarska

$$b = \frac{v_K}{3}$$

$$a = 3p_K v_K^2$$

$$R = \frac{8p_K v_K}{3T_K}$$

Dvoparametarska

	T_K / K	p_K / bar	$v_K / \text{cm}^3 \text{ mol}^{-1}$	$R / \text{J mol}^{-1} \text{ K}^{-1}$
Ar	150,86	48,98	74,57	6,456
H ₂	32,98	12,93	64,20	6,712
O ₂	154,58	50,43	73,37	6,383
H ₂ O	647,14	220,64	55,95	5,087
NH ₃	405,40	113,53	72,47	5,412
CO ₂	304,12	73,74	94,07	6,082
CH ₄	190,56	45,99	98,60	6,346
C ₂ H ₆	305,32	48,72	145,50	6,191
CO	132,85	34,94	93,10	6,530

$$a = \frac{27R^2 T_K^2}{64p_K} \quad b = \frac{RT_K}{8p_K}$$

Načelo usporedivih stanja

Reducirane veličine

$$p_r = \frac{p}{p_K}$$

$$T_r = \frac{T}{T_K}$$

$$\nu_r = \frac{\nu}{\nu_K}$$

$$z = \frac{\nu}{\nu_{id}}$$

Reducirana vdW jednadžba

$$\left(p_r + \frac{3}{\nu_r^2} \right) (3\nu_r - 1) = 8T_r$$

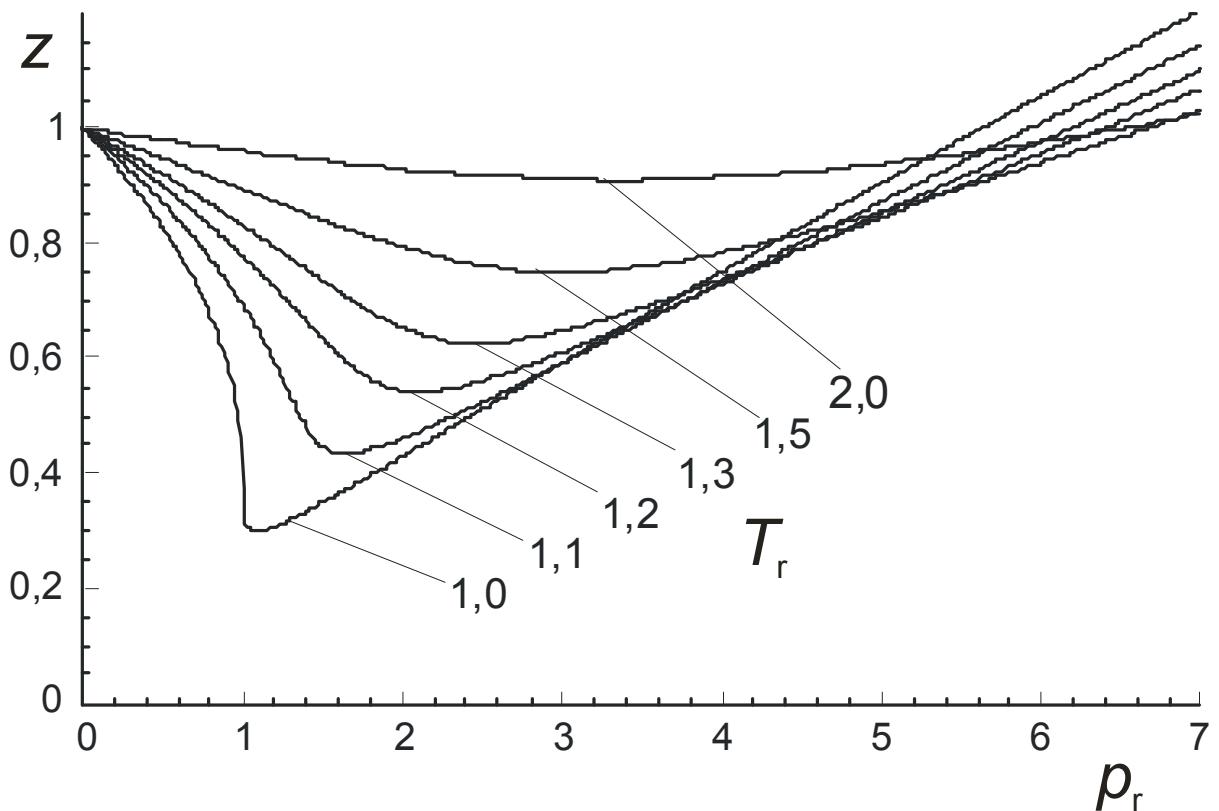
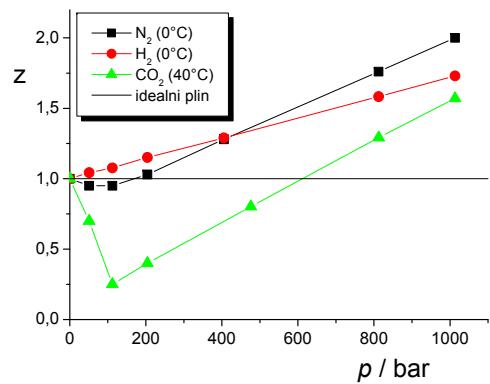
z-oblik

$$z^3 - z^2 \left(\frac{bp}{RT} + 1 \right) + z \frac{ap}{R^2 T^2} - \frac{abp^2}{R^3 T^3} = 0$$

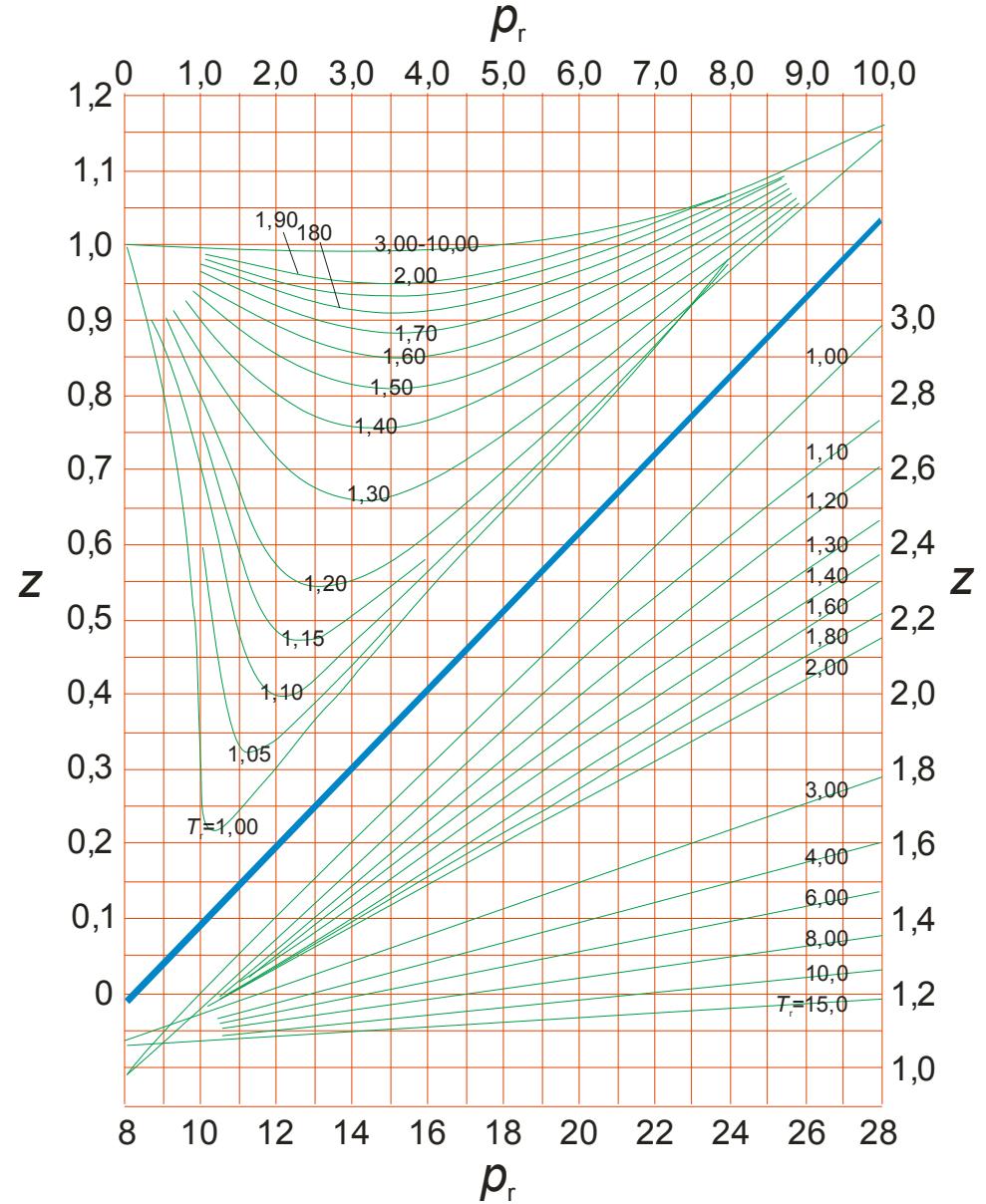
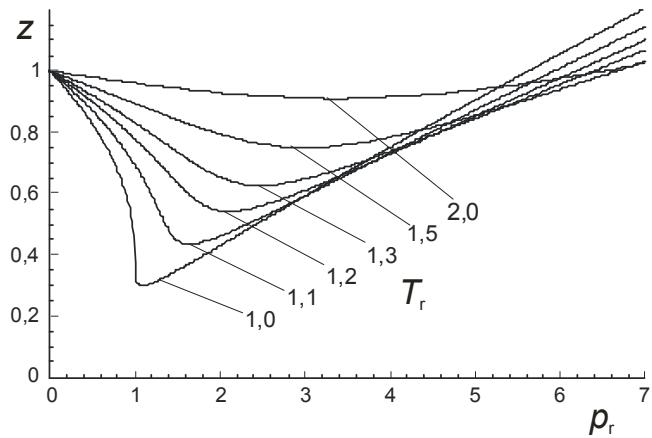
$$z^3 - z^2 \left(\frac{p_r}{8T_r} + 1 \right) + z \frac{27p_r}{64T_r^2} - \frac{27}{512} \frac{p_r^2}{T_r^3} = 0$$

Načelo usporedivih stanja

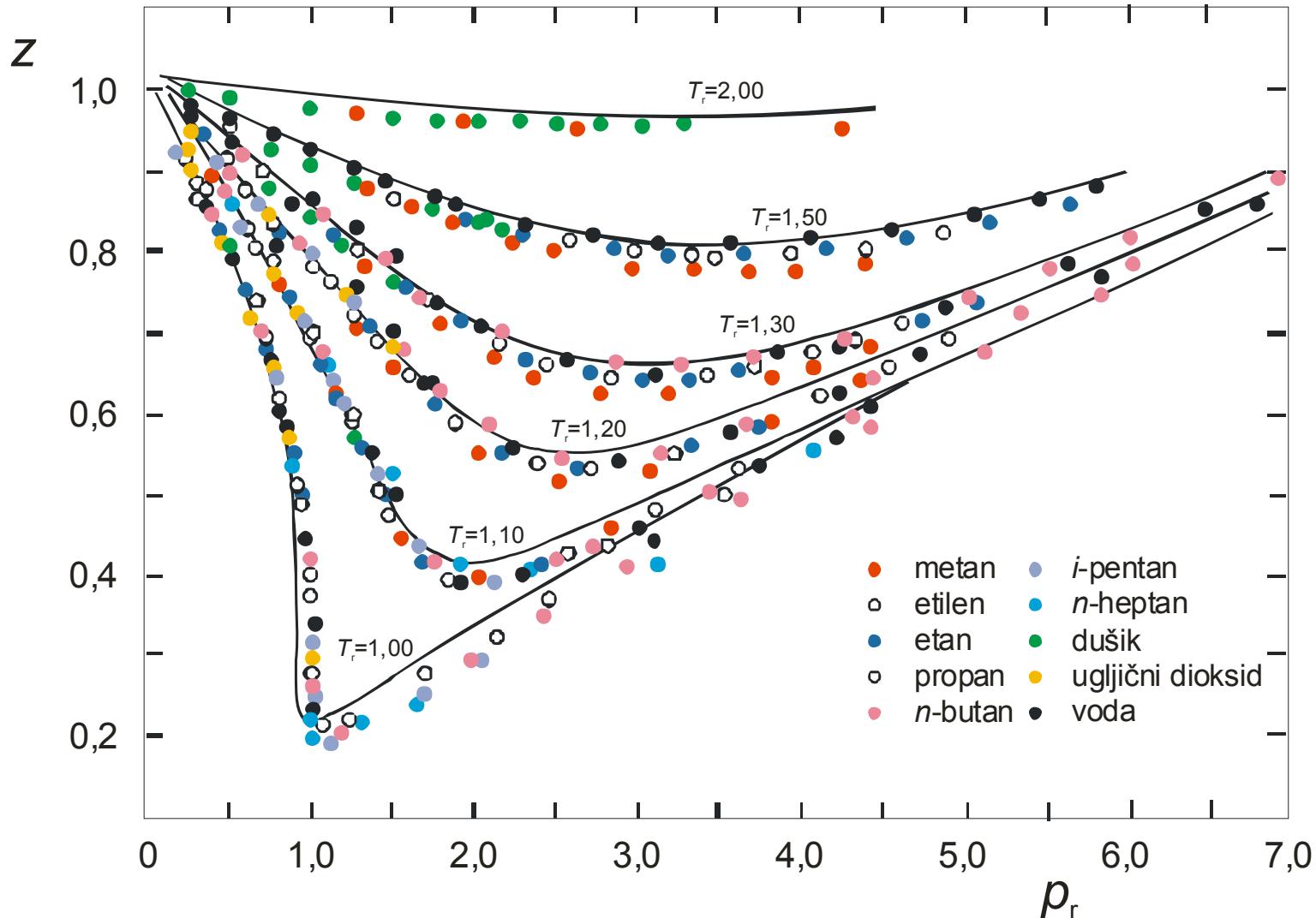
$$z = \frac{v}{v_{\text{id}}} = \frac{pv}{RT} = f(p_r, T_r)$$



Načelo usporedivih stanja



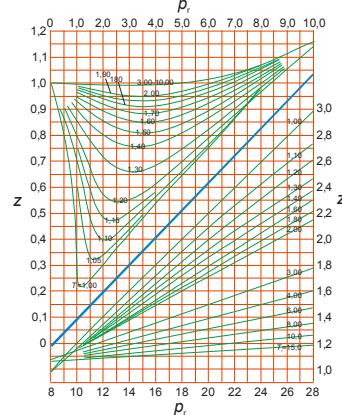
Načelo termodin. sličnosti



Načelo termodin. sličnosti

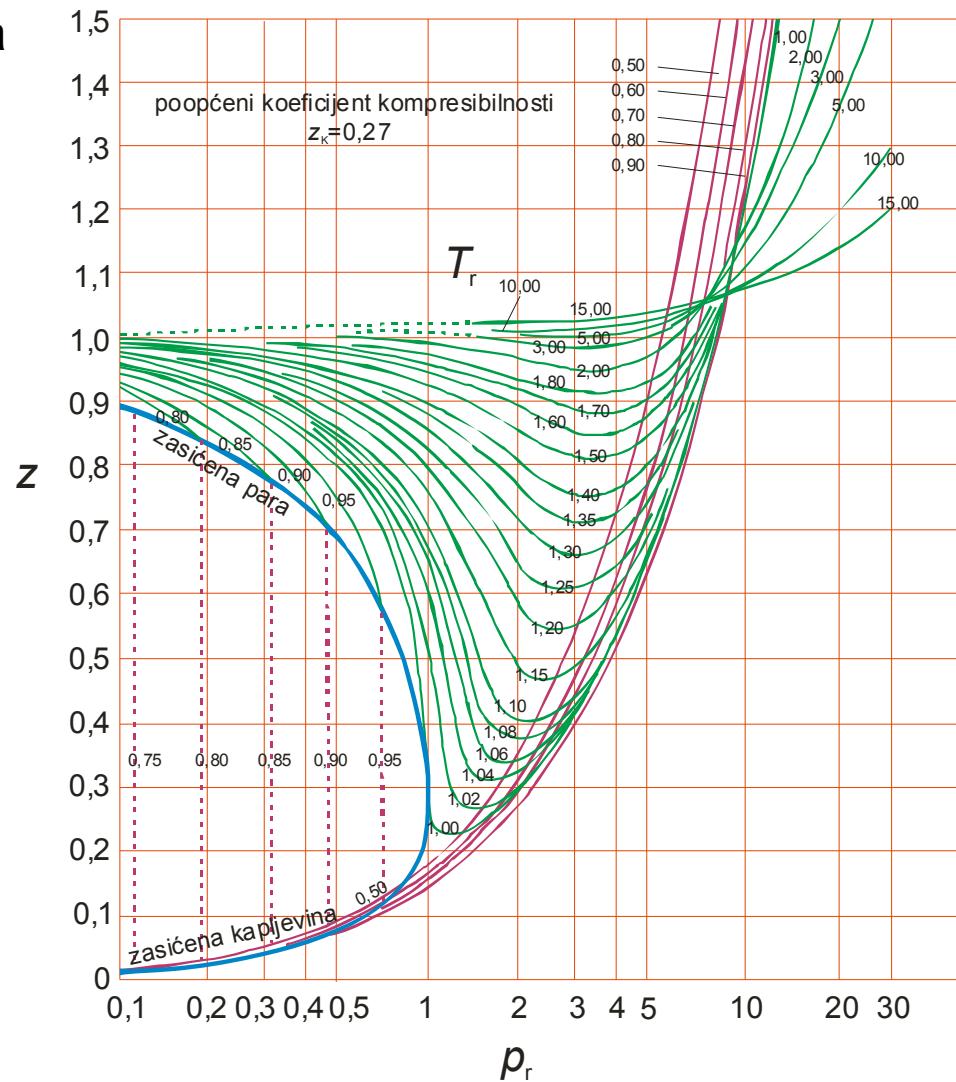
Dvoparametarska grafička korelacija

$$z = f(p_r, T_r)$$



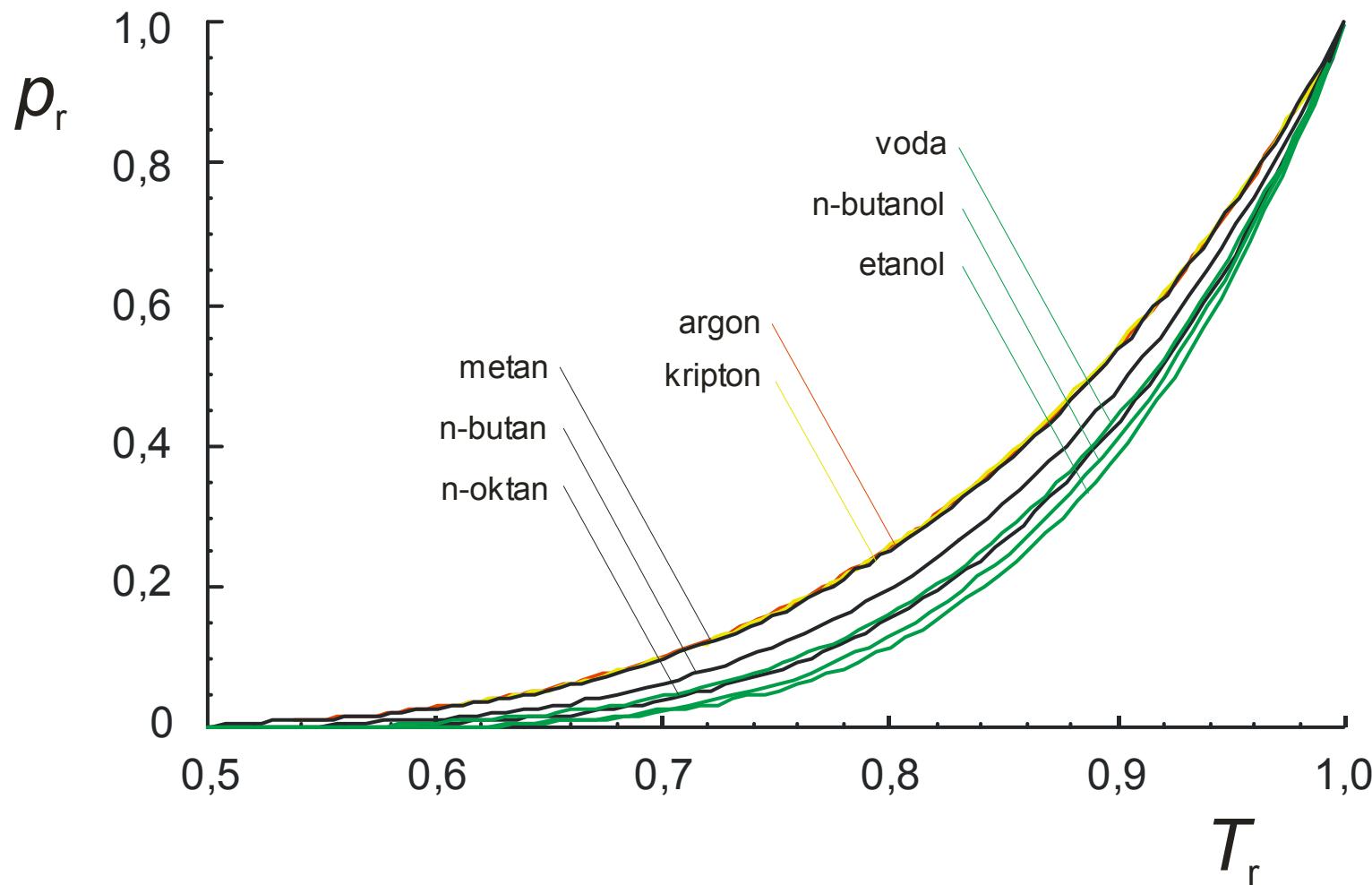
Troparametarska grafička korelacija

$$z = f(p_r, T_r, z_K)$$



Načelo termodin. sličnosti

Pitzerova korelacija (1955)



Načelo termodin. sličnosti

Pitzerova korelacija (1955)

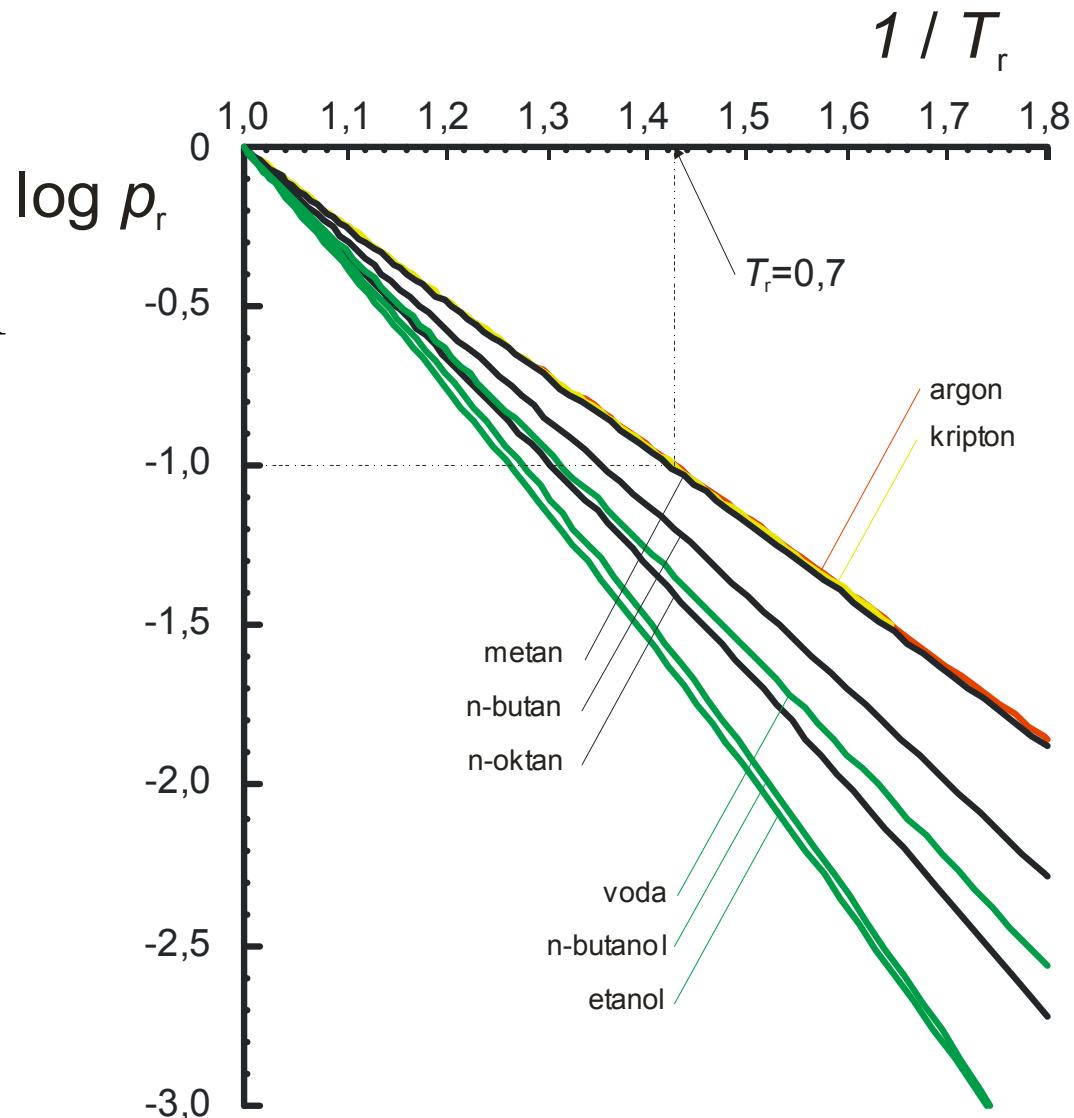
$$\omega = -\log(p_r^\bullet)_{T_r=0,7} - 1$$

Dvoparametarska korelacija

$$z = f(p_r, T_r)$$

Troparametarska korelacija

$$z = f(p_r, T_r, \omega)$$



Načelo termodin. sličnosti

Pitzerova korelacija (1955)

Taylorov red

$$z = z_{\omega=0} + \omega \left(\frac{\partial z}{\partial \omega} \right)_{\omega=0} + \frac{\omega^2}{2} \left(\frac{\partial^2 z}{\partial \omega^2} \right)_{\omega=0} + \dots$$

Linearizacija

$$z = z^{(0)} + \omega z^{(1)} + \dots$$

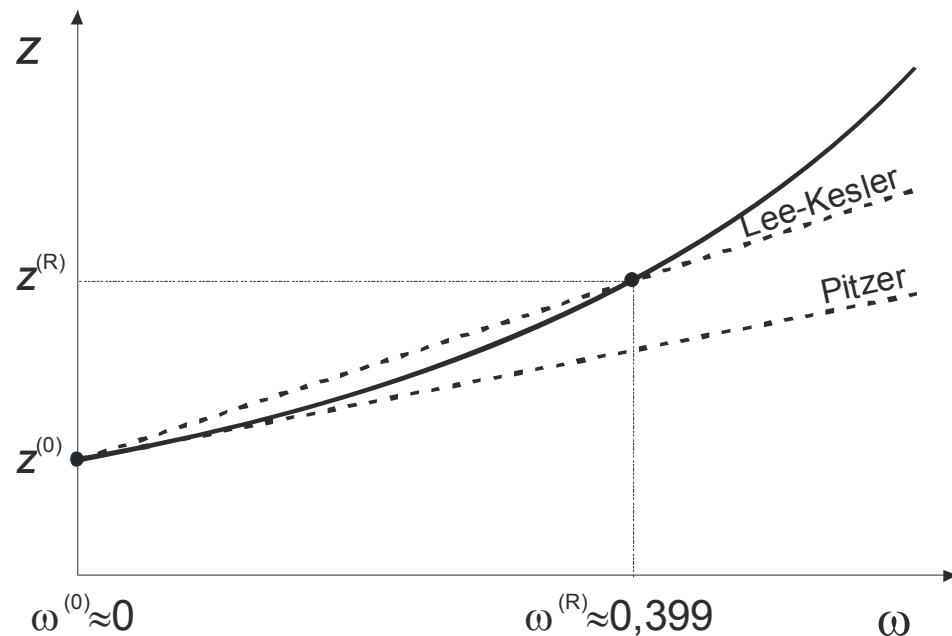
$$z = z^{(0)}(T_r, p_r) + \omega z^{(1)}(T_r, p_r)$$

$z^{(0)}$ – odsječak, $z^{(1)}$ – nagib

Načelo termodin. sličnosti

Lee-Keslerova korelacija (1975)

Dvije kapljevine
argon (sferična), $\omega=0$
 n -oktan (izdužena), $\omega=0,399$



$$\frac{z - z^{(0)}}{\omega - \omega^{(0)}} = \frac{z^{(R)} - z^{(0)}}{\omega^{(R)} - \omega^{(0)}}$$

$z^{(0)}$ – odsječak (argon)
 $z^{(R)}$ – odsječak (n -oktan)

$$z = z^{(0)} + \frac{\omega}{\omega^{(R)}} (z^{(R)} - z^{(0)})$$

$$z = z^{(0)} (T_r, p_r) + \omega z^{(1)} (T_r, p_r)$$

$z^{(0)}$ – odsječak, $z^{(1)}$ – nagib

Načelo termodin. sličnosti

	argon	<i>n</i> -oktan		argon	<i>n</i> -oktan
b_1	0,118193	0,2026579	c_3	0	0,016901
b_2	0,265728	0,331511	c_4	0,042724	0,041577
b_3	0,154790	0,027655	d_1	0,155488	0,48736
b_4	0,030323	0,203488	d_2	0,623689	0,0740336
c_1	0,236744	0,0313385	β	0,65392	1,226
c_2	0,0186984	0,0503618	γ	0,60167	0,03754

Starling BWR (SBWR), 1973

$$z = \frac{pv}{RT} = 1 + \frac{B}{v_r} + \frac{C}{v_r^2} + \frac{D}{v_r^5} + \frac{c_4}{T_r^3 v_r^2} + \left[\beta + \frac{\gamma}{v_r^2} \right] \exp \left[-\frac{\gamma}{v_r^2} \right]$$

$$B = b_1 - \frac{b_2}{T_r} - \frac{b_3}{T_r^2} - \frac{b_4}{T_r^3}$$

$$C = c_1 - \frac{c_2}{T_r} - \frac{c_3}{T_r^2}$$

$$D = d_1 - \frac{d_2}{T_r}$$

Jed. stanja trećeg stupnja

Redlich-Kwong (1949)

$$p = \frac{RT}{v-b} - \frac{a}{\sqrt{T}v(v+b)}$$

$$v^3 - \frac{RT}{p} v^2 - \left(b^2 + \frac{RTb}{p} - \frac{a}{p\sqrt{T}} \right) v - \frac{ab}{p\sqrt{T}} = 0$$

$$z^3 - z^2 - \left(\frac{b^2 p^2}{R^2 T^2} + \frac{pb}{RT} - \frac{ap}{R^2 T^2 \sqrt{T}} \right) z - \frac{abp^2}{R^3 T^3 \sqrt{T}} = 0$$

$$z^3 - z^2 + (A - B^2 - B) z - AB = 0$$

Jed. stanja trećeg stupnja

Redlich-Kwong (1949)

dvoparametarska

$$a = \frac{\Omega_a R^2 T_K^{5/2}}{p_K} \quad b = \frac{\Omega_b R T_K}{p_K}$$

Mikroskopski parametri

Makroskopski parametri

$$A = \frac{ap}{R^2 T^{5/2}} \quad B = \frac{bp}{RT}$$

Jed. stanja trećeg stupnja

Redlich-Kwong (1949)

Načelo korespondentnih stanja

$$\left[\frac{p_r}{\sqrt{T_r} \nu_r \Omega (\nu_r + \Omega)} \right] (\nu_r - \Omega) = 3T_r$$

$$z^3 - z^2 - \left(\frac{\Omega^2 p_r^2}{9T_r^2} + \frac{\Omega p_r}{3T_r} - \frac{1}{9\Omega} \frac{p_r}{T_r^{5/2}} \right) z - \frac{p_r^2}{27T_r^{7/2}} = 0$$

$$z_K(\text{RK})=1/3 \quad z_K(\text{vdW})=3/8 \quad z_K(\text{exp})=0,23-0,31$$

Jed. stanja trećeg stupnja

Soave-Redlich-Kwong (1972)

$$\textcolor{red}{p} = \frac{RT}{v - b} - \frac{a\alpha}{v(v + b)}$$

$$\textcolor{red}{v}^3 - \frac{RT}{p} \textcolor{red}{v}^2 - \left(b^2 + \frac{RTb}{p} - \frac{a\alpha}{p} \right) \textcolor{red}{v} - \frac{a\alpha b}{p} = 0$$

$$z^3 - \textcolor{red}{z}^2 - \left(\frac{b^2 p^2}{R^2 T^2} + \frac{bp}{RT} - \frac{a\alpha p}{R^2 T^2} \right) z - \frac{a\alpha b p^2}{R^3 T^3} = 0$$

$$\textcolor{red}{z}^3 - \textcolor{red}{z}^2 + (A - B^2 - B) \textcolor{red}{z} - AB = 0$$

Jed. stanja trećeg stupnja

Soave-Redlich-Kwong (1972)

$$a = \frac{\Omega_a R^2 T_K^2}{p_K} \quad b = \frac{\Omega_b R T_K}{p_K}$$

$$\alpha = \left[1 + \kappa \left(1 - \sqrt{T_r} \right) \right]^2$$

$$\kappa = 0,48508 + 1,55171\omega - 0,15613\omega^2$$

$$A = \frac{a\alpha p}{R^2 T^2}$$

$$B = \frac{bp}{RT}$$

Jed. stanja trećeg stupnja

Peng-Robinson (1976)

$$p = \frac{RT}{v-b} - \frac{a\alpha}{v^2 + 2bv - b^2}$$

$$v^3 - \left(\frac{RT}{p} - b \right) v^2 - \left(3b^2 + \frac{2RTb}{p} - \frac{a\alpha}{p} \right) v - \left(\frac{a\alpha b}{p} - \frac{RTb^2}{p} - b^3 \right) = 0$$

$$z^3 - \left(1 - \frac{bp}{RT} \right) z^2 - \left(\frac{3b^2 p^2}{R^2 T^2} + \frac{2bp}{RT} - \frac{a\alpha p}{R^2 T^2} \right) z - \left(\frac{a\alpha bp^2}{R^3 T^3} - \frac{b^2 p^2}{R^2 T^2} - \frac{b^3 p^3}{R^3 T^3} \right) = 0$$

$$z^3 - (1-B)z^2 + (A-3B^2-2B)z - (AB-B^2-B^3) = 0$$

Jed. stanja trećeg stupnja

Peng-Robinson (1976)

$$a = \frac{\Omega_a R^2 T_K^2}{p_K} \quad b = \frac{\Omega_b R T_K}{p_K}$$

$$\alpha = \left[1 + \kappa \left(1 - \sqrt{T_r} \right) \right]^2$$

$$A = \frac{a\alpha p}{R^2 T^2}$$

$$\kappa = 0,37464 + 1,54226 \omega - 0,26992 \omega^2$$

$$B = \frac{bp}{RT}$$

$$z_K(\text{PR})=0,3074$$

Jed. stanja trećeg stupnja

Clausius

$$p = \frac{RT}{v - b} - \frac{a}{T(v + c)^2}$$

Berthelot

$$p = \frac{RT}{v - b} - \frac{a}{Tv^2}$$

Patel-Teja

$$p = \frac{RT}{v - b} - \frac{a\alpha}{v(v + b) + c(v - b)}$$

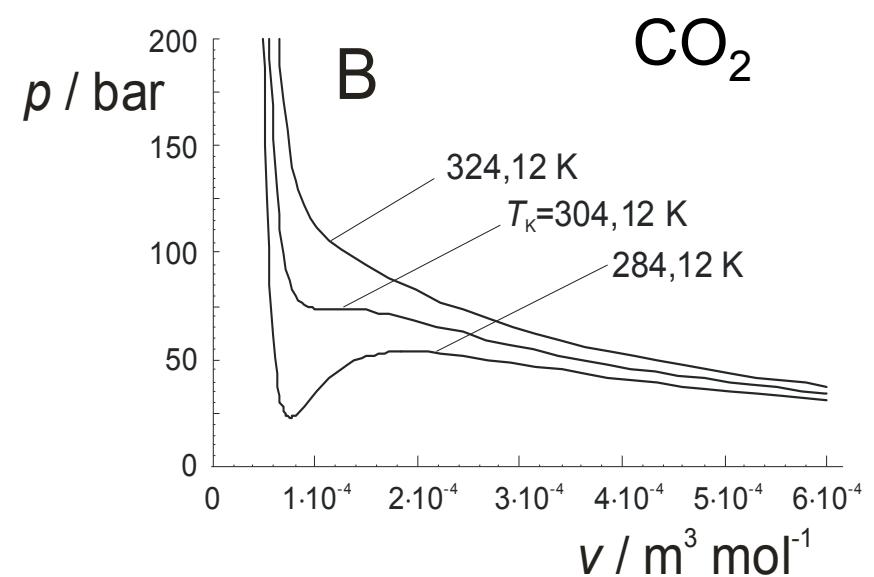
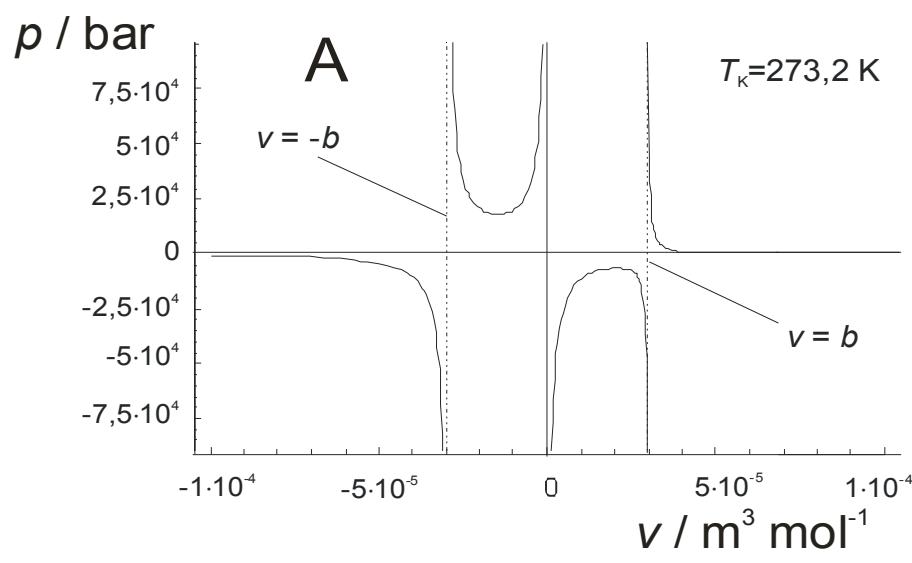
Peng-Robinson-Stryjek-Vera

Treble-Bishnoi

$$p = \frac{RT}{v - b} - \frac{a\alpha}{v^2 + (b + c)v - bc + d^2}$$

Jed. stanja trećeg stupnja

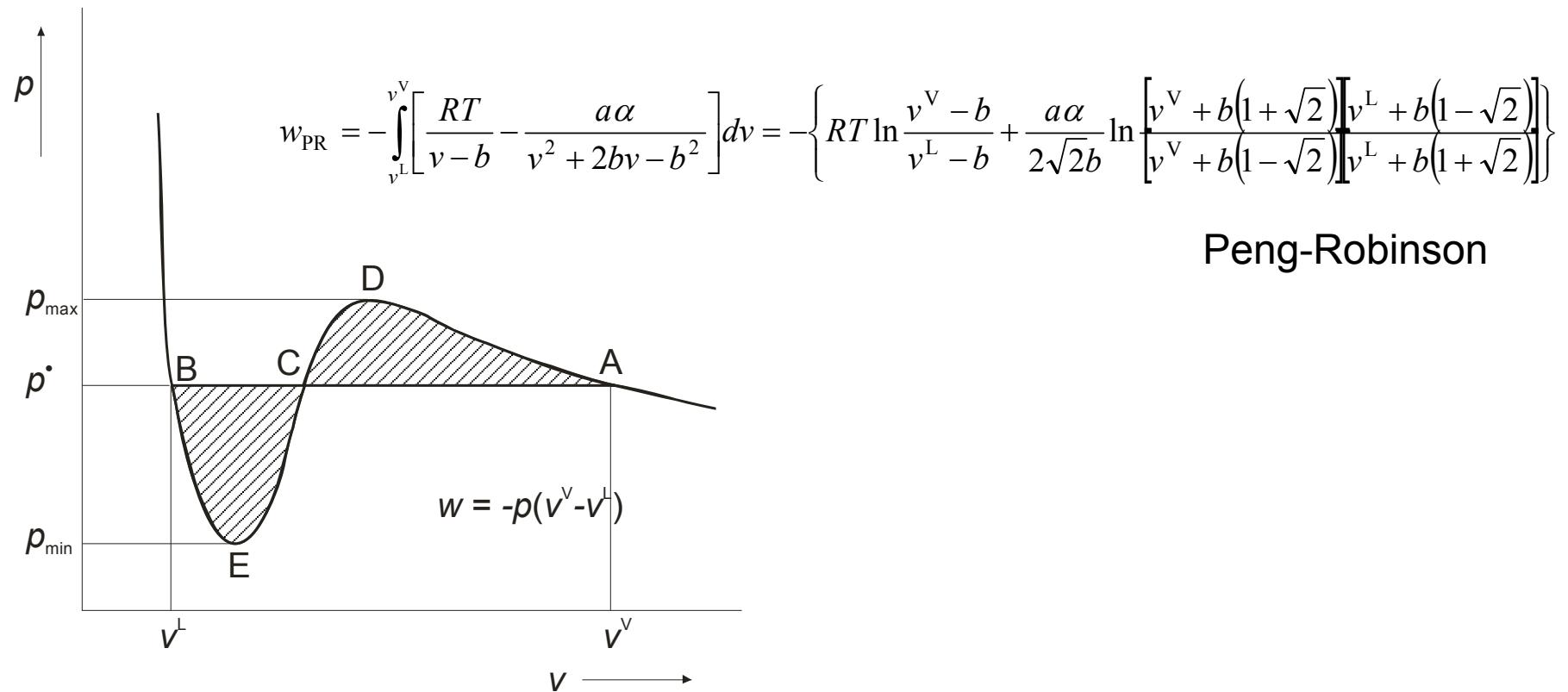
Izračunavanje volumetrijskih svojstava



Jed. stanja trećeg stupnja

Izračunavanje volumetrijskih svojstava

Izbor stabilne faze



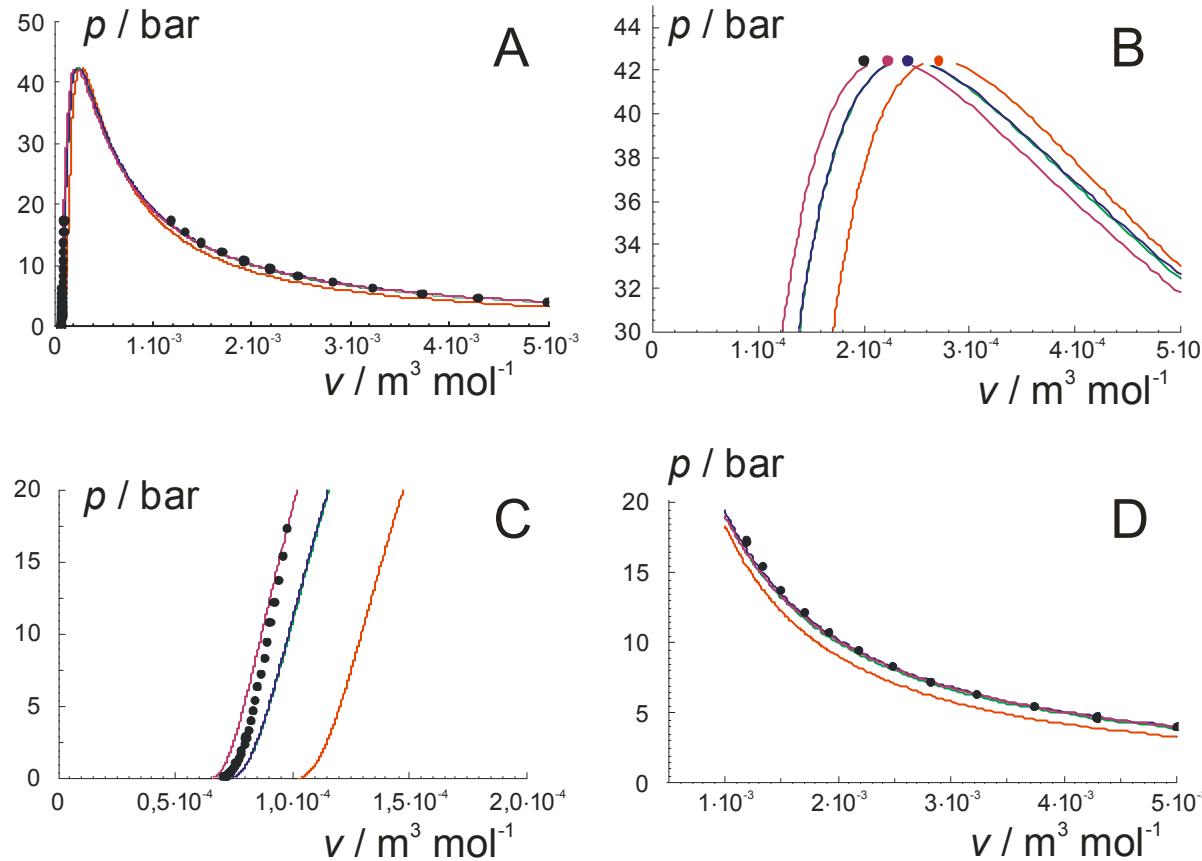
Usporedba jednadžbi stanja

NAZIV	FORMALNI PRIKAZ	PARAMETRI
Opća plinska jednadžba	$pV = RT$	-
van der Waals	$\left[p + \frac{a}{v^2} \right] (v - b) = RT$	T_K, p_K
Virijalna jednadžba stanja, dvočlana	$p[v - B(T)] = RT$	$T_K, p_K(v_K), \omega, \mu, a_3, \dots$
Virijalna jednadžba stanja, tročlana	$\left[p - \frac{RTvB(T) + RTC(T)}{v^3} \right] v = RT$	$T_K, p_K(v_K), \omega, \mu, a_3, \dots$
Clausius	$\left[p + \frac{a}{T(v+c)^2} \right] (v - b) = RT$	T_K, p_K, v_K
Berthelot	$\left(p + \frac{a}{Tv^2} \right) (v - b) = RT$	a, b
Redlich-Kwong	$\left[p + \frac{a}{\sqrt{Tv}(v+b)} \right] (v - b) = RT$	T_K, p_K

Usporedba jednadžbi stanja

NAZIV	FORMALNI PRIKAZ	PARAMETRI
Soave-Redlich-Kwong	$\left[p + \frac{a\alpha(T)}{v(v+b)} \right] (v-b) = RT$	T_K, p_K, ω
Peng-Robinson	$\left[p + \frac{a\alpha(T)}{v(v+b)+b(v-b)} \right] (v-b) = RT$	T_K, p_K, ω
Patel-Teja	$\left[p + \frac{a\alpha(T)}{v(v+b)+c(v-b)} \right] (v-b) = RT$	T_K, p_K, ω
Stryjek-Vera-Peng-Robinson	$\left[p + \frac{a\alpha(T)}{v(v+b)+b(v-b)} \right] (v-b) = RT$	T_K, p_K, ω, K_1
Treble-Bishnoi	$\left\{ p + \frac{a\alpha(T)}{v^2 + [b(T)+c]v - b(T)c + d^2} \right\} [v - b(T)] = RT$	$T_K, p_K, z_K, (v_K), \omega$

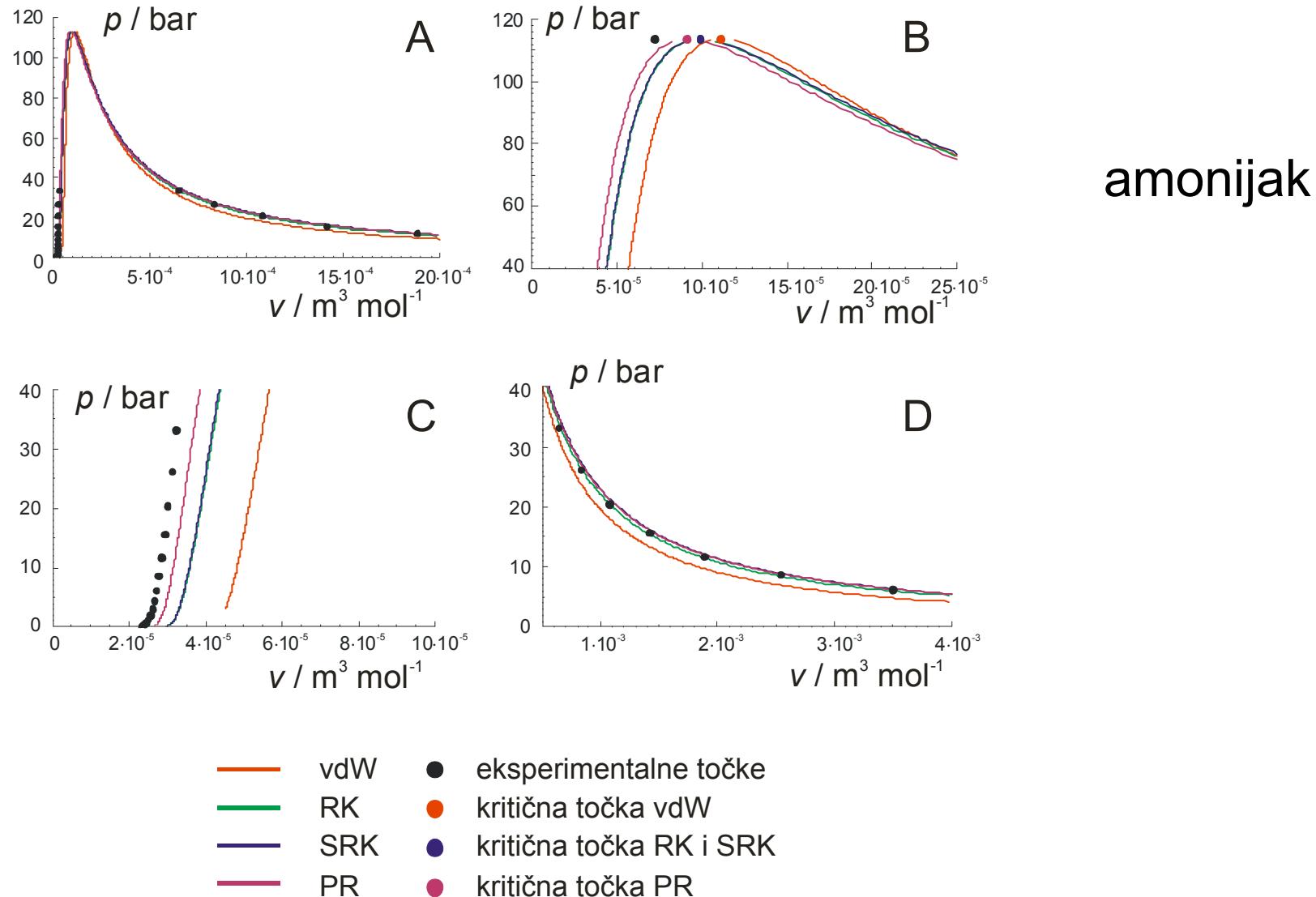
Usporedba jednadžbi stanja



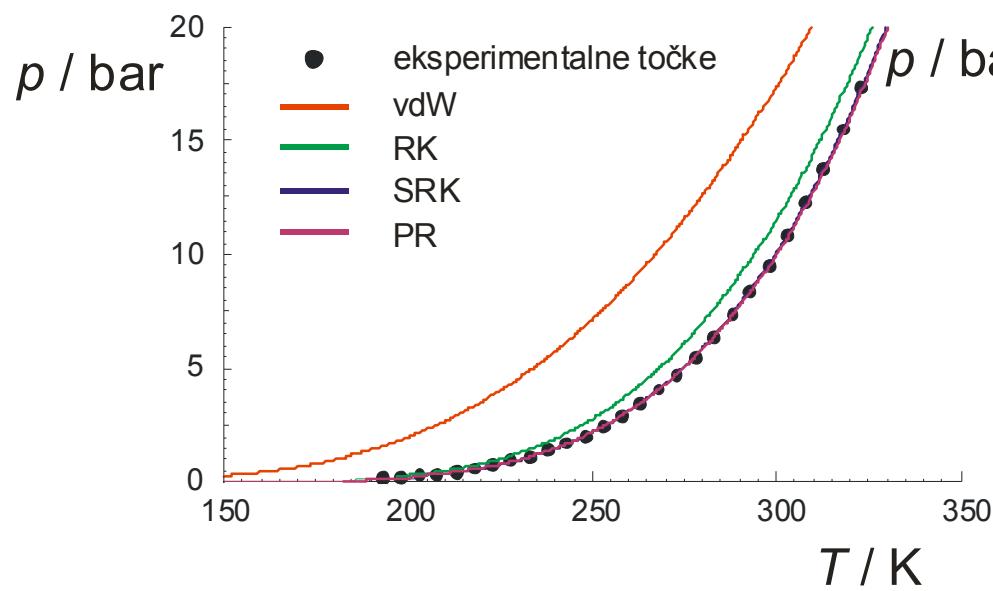
propan

- vdW ● eksperimentalne točke
- RK ● kritična točka vdW
- SRK ● kritična točka RK i SRK
- PR ● kritična točka PR

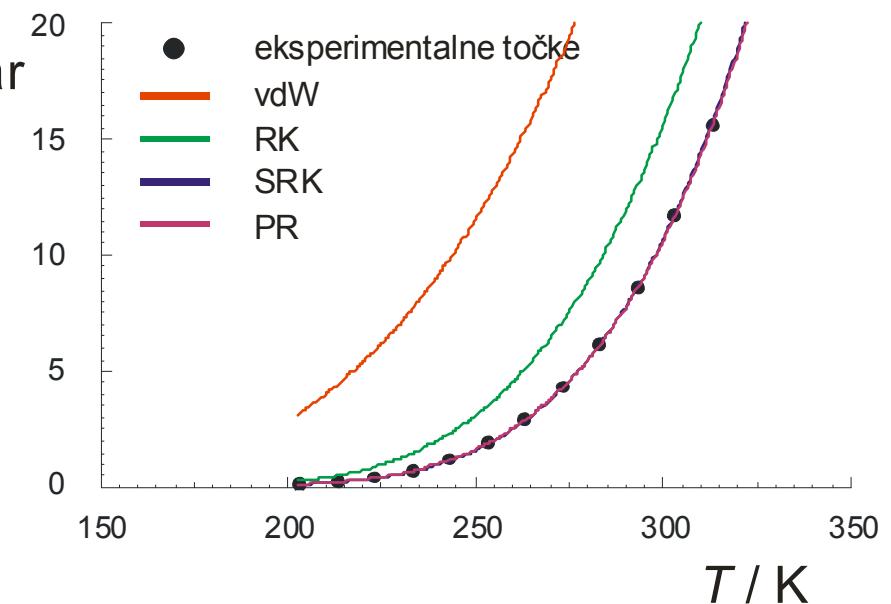
Usporedba jednadžbi stanja



Usporedba jednadžbi stanja

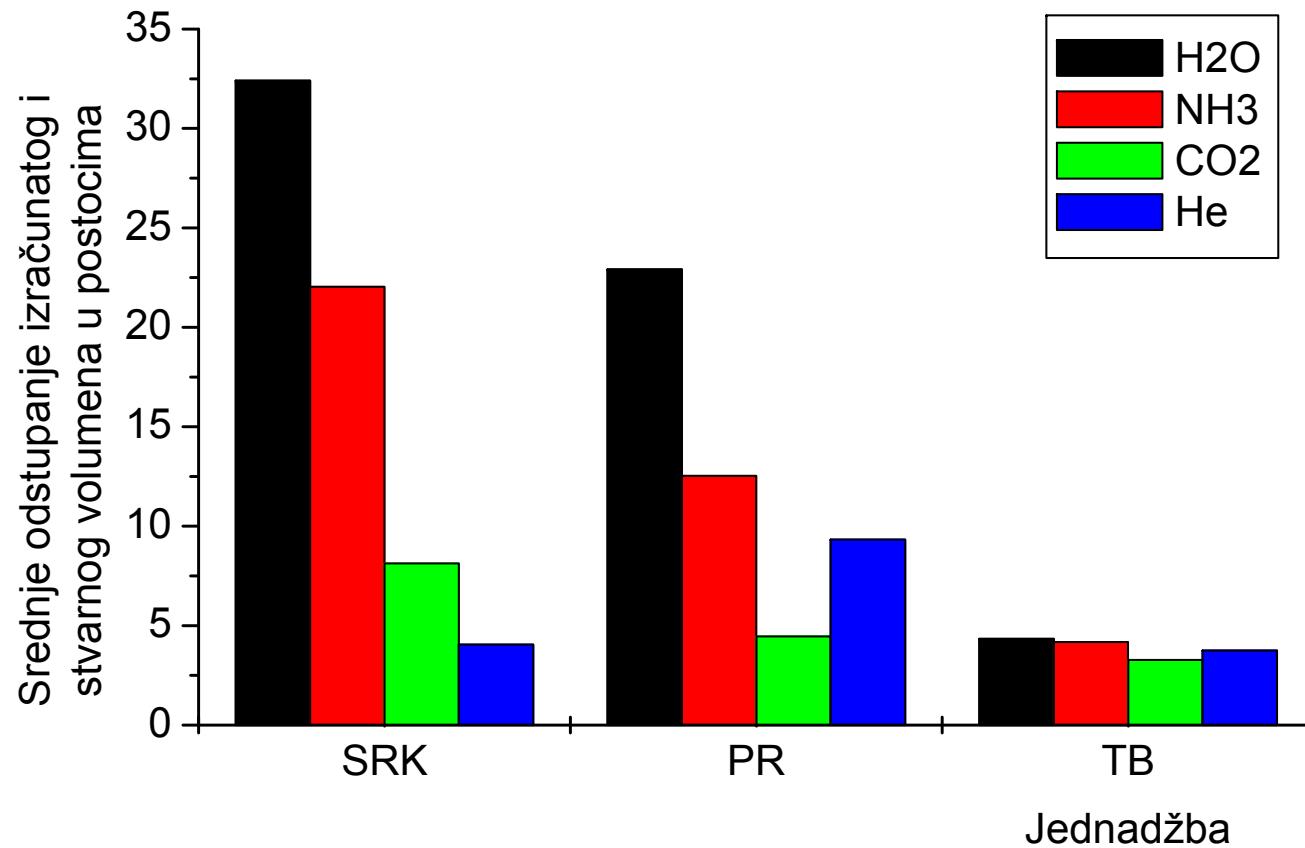


propan



amonijak

Usporedba jednadžbi stanja



Plinske smješe

Redlich-Kwong jednadžba
stanja za smjesu:

$$p = \frac{RT}{v - b_M} - \frac{a_M}{\sqrt{T}v(v + b_M)}$$

PRAVILA MIJEŠANJA

Parametri jednadžbe
stanja za smjesu:

$$a_M = f(a_i)$$
$$b_M = f(b_i)$$

Parametri jednadžbe
stanja za smjesu:

$$a_M = \frac{\Omega_a R^2 T_{KM}^{5/2}}{p_{KM}}$$
$$b_M = \frac{\Omega_b R T_{KM}}{p_{KM}}$$

Parametri jednadžbe
stanja za komponente:

$$a_i = \frac{\Omega_a R^2 T_K^{5/2}}{p_K}$$
$$b_i = \frac{\Omega_b R T_K}{p_K}$$

Kritični parametri
za smjesu:

$$T_{KM} = f(T_{Ki})$$
$$p_{KM} = f(p_{Ki})$$

PSEUDOKRITIČNI PARAMETRI

Eksperimentalni podaci:
kritični parametri komponenata

$$p_{Ki}, T_{Ki}$$

Plinske smjese

Pseudokritični parametri

Van der Waals

$$v_{KM} = \sum y_i v_{Ki}$$

$$\frac{T_{KM}}{\sqrt{p_{KM}}} = \sum \frac{y_i T_{Ki}}{\sqrt{p_{Ki}}}$$

$$\frac{T_{KM}}{p_{KM}} = \sum \frac{y_i T_{Ki}}{p_{Ki}}$$

Kay (1930)

$$v_{KM} = \sum y_i v_{Ki}$$

$$T_{KM} = \sum y_i T_{Ki}$$

$$p_{KM} = \sum y_i p_{Ki}$$

Prausnitz-Gunn (1958)

$$v_{KM} = \sum y_i v_{Ki}$$

$$z_{KM} = \sum y_i z_{Ki}$$

$$T_{KM} = \sum y_i T_{Ki}$$

$$p_{KM} = \frac{z_{KM} RT_{KM}}{v_{KM}}$$

Plinske smješe

Pravila miješanja

RK, SRK, PR

$$b_M = \sum y_i b_i$$

$$B_M = \sum y_i B_i$$

$$b_M = y_1 b_1 + y_2 b_2$$

$$(a\alpha)_M = \sum \sum y_i y_j (a\alpha)_{ij}$$

$$A_M = \sum \sum y_i y_j A_{ij}$$

$$a_M = \sum \sum y_i y_j a_{ij} \quad \text{RK}$$

$$a_M = y_1^2 a_1 + 2y_1 y_2 a_{12} + y_2^2 a_2$$

$$a_{ij} = \sqrt{a_i a_j} \quad \text{RK}$$

$$a_{ij} = (1 - k_{ij}) \sqrt{a_i a_j} \quad \text{SRK, PR}$$